LINEAR ALGEBRA

new problems for group 1 and group 2, November 11-th

Changed because of an error noticed by a student November 12-th, 4.35 a.m.

There will be 45–minutes test on November 17 so less than a week from today (November 11) There will be problems based on what we were doing since October. You should know how to solve systems of linear equations, how to check if some vectors are linearly independent, how to check if the set of vectors is a basis of a linear space.

Remainders:

1. A vector space V is a set consisting of some elements that can be added and multiplied by numbers.

2. A linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ with coefficients $c_1, c_2, \ldots, c_n \in \mathbb{R}$ is a sum $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$.
   The set of all linear combinations of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is denoted by $\mathrm{Lin}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.
   If $V = \mathrm{Lin}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span the space V.

3. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ are linearly independent if and only if the only linear combination of them equal to $\mathbf{0}$ is the one with all coefficients equal to 0.

4. A basis of V is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that each vector $\mathbf{v} \in V$ can be written in exactly one way as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
   It easy to see that the elements of a basis are linearly independent. All bases of a vector space V consist of the same number of elements.
   The number of the elements of a basis of V is called **a dimension** V**.** It is denoted $\dim V$.
   If vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ are linearly independent and $k < \dim V$ then there exist vectors $\mathbf{w}_{k+1}, \mathbf{w}_{k+2}, \ldots, \mathbf{w}_n$, $n = \dim V$ such that the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of V.

70. Let $\mathbf{v}_1 = (3, 2, 1), \mathbf{v}_2 = (5, 4, 1)$. Prove that vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. For which $p \in \mathbb{R}$ the vectors $\mathbf{v}_1, \mathbf{v}_2, (3, p, 1)$ are linearly dependent?

Find an equation of the plane through $(0, 0, 0)$, $(3, 2, 1)$ and $(5, 4, 1)$.

Solution. If $c_1(3, 2, 1) + c_2(5, 4, 1) = (0, 0, 0)$ then $c_1 + c_2 = 0$ (the third coordinate) then $0 = 3c_1 + 5c_2 = 2c_2$, so $c_2 = 0$ and $0 = c_1 + c_2 = c_1$. The proof is done. One can write it using matrices:

$$\begin{pmatrix} 3 & 5 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we shall look at the three vectors

$$\begin{pmatrix} 3 & 5 & 3 \\ 2 & 4 & p \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & p-2 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p-2 \\ 1 & 0 & 1 \end{pmatrix}$$

If $p = 2$. the the vectors are linearly dependent because two of them coincide. If $p \neq 2$ then they are linearly independent because further reduction is possible and we get the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

A general equation of the plane in the three dimensional space is $ax + by + cz + d = 0$. Since the point $(0,0,0)$ lies on the plane we have $0 = a \cdot 0 + b \cdot 0 + c \cdot 0 + d = d$. Therefore the equation is $0 = ax + by + cz$. Therefore $0 = 3a + 2b + c$ and $0 = 5a + 4b + c$. We have a system of two linear homogeneous equations with the three unknowns. The matrix is $\begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, so $a + c = 0 = a + b$ so $b = c = -a$. An equation of the plane is therefore $ax - ay - az = 0$ with $a \neq 0$. Little simplification gives us $x - y - z = 0$. We are done. $\square$

71. Let $V = \mathrm{Lin}\big((1,-2,1,2,-1),(2,1,2,2,1),(4,7,4,2,5)\big) \subset \mathbb{R}^5$.
Find a system of linear equations with the set of solutions equals V
For what t the vector $(t^2 - 2, 2 + 2t, 7, 10, -1) \in V$?
Complete the set $\{(1,-2,1,2,-1),(2,1,2,2,1),(0,0,0,0,1)\}$ to a basis of $\mathbb{R}^5$

Solution. $\begin{pmatrix} 1 & -2 & 1 & 2 & -1 \\ 2 & 1 & 2 & 2 & 1 \\ 4 & 7 & 4 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 2 & -1 \\ 0 & 5 & 0 & -2 & 3 \\ 0 & 15 & 0 & -6 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 2 & -1 \\ 0 & 5 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to$
$\begin{pmatrix} 5 & 0 & 5 & 6 & 1 \\ 0 & 5 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ The result tells us that if $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are rows of the initial matrix then $3(\mathbf{r}_2 - 2\mathbf{r}_1) = \mathbf{r}_3 - 4\mathbf{r}_1$. This may be written as $-2\mathbf{r}_1 + 3\mathbf{r}_2 - \mathbf{r}_3 = \mathbf{0}$. The three rows are linearly dependent. The rows $\mathbf{r}_1, \mathbf{r}_2$ are linearly independent, notice that $\mathbf{r}_3$ was not used for changing of the first or the second row. This proves that the dimension of the space V is 2.

We need three independent equations of the type $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$ to define V as a set of solutions to the system of equations. These equations must be satisfied by the points $(5,0,5,6,1) \in V$ and $(0,5,0,-2,3) \in V$. This can be done as follows. Let us choose three linearly independent vectors in the three dimensional space with coordinates a_3, a_4, a_5. For example $(a_3,a_4,a_5) = (1,0,0)$, $(a_3,a_4,a_5) = (0,5,0)$ and $(a_3,a_4,a_5) = (0,0,5)$. We obtain the solutions of the system of two linear homogenous equations $(-1,0,1,0,0)$, $(-6,2,0,5,0)$ and $(-1,-3,0,0,5)$. So the three equations are $-x_1 + x_3 = 0$, $-6x_1 + 2x_2 + 5x_4 = 0$ and $-x_1 - 3x_2 + 5x_5 = 0$.
There are infinitely many different choices of the three equations. We may start with another basis of $\mathbb{R}^3$ e.g. $(a_3,a_4,a_5) = (7,5,5)$, $(a_3,a_4,a_5) = (1,-5,5)$ and $(a_3,a_4,a_5) = (1,1,-1)$.

Then the corresponding five-tuples are $(-14,-1,7,5,5)$, $(4,-5,1,-5,5)$, $(-2,1,1,1,-1)$ and in this case the three equations are $-14x_1-x_2+7x_3+5x_4+5x_5=0$, $4x_1-5x_2+x_3-5x_4+5x_5=0$ and $-2x_1+x_2+x_3+x_4-x_5=0$.

Now we have to find all $t\in\mathbb{R}$ for which $(t^2-2,2+2t,7,10,-1)\in V$. The question is: for what t the equation $0=-x_1+x_3=-t^2+2+7=-t^2+9$ holds. This happens for $t=\pm3$ so the vectors in question are $(7,8,7,10,-1)$ and $(7,-4,7,10,-1)$. The first one satisfies neither $-6x_1+2x_2+5x_4=0$ nor $-x_1-3x_2+5x_5=0$. The second satisfies all three.

The last part of this problem is to find numbers $x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3,y_4,y_5$ such that the rows of the matrix $\begin{pmatrix}1&-2&1&2&-1\\2&1&2&2&1\\0&0&0&0&1\\x_1&x_2&x_3&x_4&x_5\\y_1&y_2&y_3&y_4&y_5\end{pmatrix}$ will be linearly independent. Since the vectors $(1,-2)$ and $(2,1)$ are linearly independent we may try to work with $x_1=x_2=y_1=y_2=0$.

Under this hypothesis the matrix is $\begin{pmatrix}1&-2&1&2&-1\\2&1&2&2&1\\0&0&0&0&1\\0&0&x_3&x_4&x_5\\0&0&y_3&y_4&y_5\end{pmatrix}$. As everybody knows the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are linearly independent. This easily implies that the rows of the matrix $\begin{pmatrix}1&-2&1&2&-1\\2&1&2&2&1\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}$ are linearly independent:

$$\begin{pmatrix}1&-2&1&2&-1\\2&1&2&2&1\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}\to\begin{pmatrix}1&-2&0&0&0\\2&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}\to\begin{pmatrix}1&-2&0&0&0\\0&5&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}\to$$

$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}$. Also in this case there are infinitely many correct answers. $\square$

A solution to problem 17 from ALWNEcw.pdf

Consider a subspace $W \subset \mathbb{R}^5$ described by the following system of equations:

$$\begin{cases} x_1 & - & x_2 & + & 2x_3 & - & x_4 & + & x_5 & = & 0, \\ 2x_1 & + & 3x_2 & - & x_3 & + & 2x_4 & - & x_5 & = & 0. \end{cases}$$

a) Find a basis of W. Complete the basis to a basis of the space $\mathbb{R}^5$.

b) Complete the system $\mathbf{v}_1 = (1,0,0,1,0)$, $\mathbf{v}_2 = (0,1,0,2,1)$, $\mathbf{v}_3 = (1,0,0,1,1)$ to a basis of the space $\mathbb{R}^5$, using some vectors of W.

Solution. The matrix of the system is $\begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 2 & 3 & -1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 5 & -5 & 4 & -3 \end{pmatrix} \to$ $\begin{pmatrix} 5 & 0 & 5 & -1 & 2 \\ 0 & 5 & -5 & 4 & -3 \end{pmatrix}$. The solutions are fully detemined by the values of the unknowns x_3, x_4, x_5, It suffices to choose three linearly independent vectors in $\mathbb{R}^3$ and evaluate the values of x_1, x_2 for them. Let us start with $(1,0,0)$. We obtain $\mathbf{w}_1 = (-1,1,1,0,0)$. Now let us look at $(0,5,0)$. We get $\mathbf{w}_2 = (1,-4,0,5,0)$. The last we need is $(0,0,5)$ and the result in this case is $\mathbf{w}_3 = (-2,3,0,0,5)$. A basis of W we have found is $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. It is clear that there are infinitely many other bases. Let $\mathbf{w}_4 = (1,0,0,0,0)$ and $\mathbf{w}_5 = (0,1,0,0,0)$. It is very easy to see that the vectors $\mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent, just look at the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & -4 & 0 & 5 & 0 \\ -2 & 3 & 0 & 0 & 5 \end{pmatrix}$. There are zeros above the diagonal, there is no zero on the diagonal. In such a situation the rows are linearly independent, it follows immediately from the definition of the linear independence.

Now let us solve part b. We start from simplifying the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The rows of the matrix are linearly independent, the third entry of each of them is 0. One can see that to them the vectors $\mathbf{w}_2$ and $\mathbf{w}_1$ gives us five vectors which are linearly independent (a proof later on), so it is a basis of $\mathbb{R}^5$. The matrix is $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -4 & 0 & 5 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix}$. Now we prove the linear independence of the rows. Let c_1, c_2, c_3, c_4, c_5 be such numbers that the linear combination of the rows with them as the coefficients is $(0,0,0,0,0)$. This implies that $c_5 = 0$ (look at the third component). $c_3 = 0$ (the last component). $c_1 + c_4 = 0$ (the first component), $c_2 - 4c_4 = 0$ (the second component). Then $0 = c_1 + 2c_2 + 5c_4 = -c_4 + 8c_4 + 5c_4 = 12c_4$ so $c_4 = 0$ and therefore also $c_1 = 0$ and $c_2 = 0$.

We were lucky to be able to complete the given vectors to a basis of $\mathbb{R}^5$ with the vectors from the basis of SW found before. Not everybody and not always has luck. If we do not see thew vectors in WW that could be added to the given ones we can say that we want to add vectors $\mathbf{v}_4 = a\mathbf{w}_1 + b\mathbf{w}_2 + cw_3$ and $\mathbf{v}_5 = p\mathbf{w}_1 + q\mathbf{w}_2 + rw_3$. Then simplify the matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ to see how to choose the numbers a, b, c, p, q, r to quarantee the linear independence of the five vectors. The initial matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ -a+b-2c & a-4b+3c & a & 5b & 5c \\ -p+q-2r & p-4q+3r & p & 5q & 5r \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a+b-2c & a-4b+3c & a & 5b & 0 \\ -p+q-2r & p-4q+3r & p & 5q & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a & -a+12b-4c & 0 \\ 0 & 0 & p & -p+12q-4r & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & a & -a+12b-4c & 0 \\ 0 & 0 & p & -p+12q-4r & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now it is enough to set $p = 0 \neq a$ and $12q - 4r \neq 0$ Obviously it is possible, e.g. $a = 1$, $q = 1$ and $r = 0$. This way you can see a solution less dependent on someone's guess. $\square$

I may add few more problems in the nearest future. Try to solve them BEFORE the test planned for November 17^{th}.

72. Solve systems of linear equations:

$$\begin{cases} 2x - y - z = 4 \\ 3x + 4y - 2z = 11 \\ 3x - 2y + 4z = 11 \end{cases} \quad \begin{cases} x + y + 2z = -1 \\ 2x - y + 2z = -4 \\ 4x + y + 4z = -2 \end{cases} \quad \begin{cases} 3x + 2y + z = 5 \\ 2x + 3y + z = 1 \\ 2x + y + 3z = 11 \end{cases} \quad \begin{cases} x + 2y + 4z = 31 \\ 5x + y + 2z = 29 \\ 3x - y + z = 10 \end{cases}$$

$$\begin{cases} w + x + 2y + 3z = 1 \\ 3w - x - y - 2z = -4 \\ 2w + 3x - y - z = -6 \\ w + 2x + 3y - z = -4 \end{cases} \quad \begin{cases} w + 2x + 3y - 2z = 6 \\ 2w - x - 2y - 3z = 8 \\ 3w + 2x - y + 2z = 4 \\ 2w - 3x + 2y + z = -8 \end{cases} \quad \begin{cases} w + 2x + 3y + 4z = 5 \\ 2w + x + 2y + 3z = 1 \\ 3w + 2x + y + 2z = 1 \\ 4w + 3x + 2y + z = -5 \end{cases}$$