Linear Algebra

new problems for group 1 and group 2

60. Is the set $\{(1,2,3,4),(5,6,7,8)\}$ a basis of $\mathbb{R}^4$?

Solution. It is not because each basis of $\mathbb{R}^4$ consists of 4 elements. $\square$

61. Which of the following sets are basis of $\mathbb{R}^3$:

$\{(1,1,1),(1,2,1),(2,1,1),\}$, $\{(2,3,2),(4,0,2),(-2,3,0)\}$, $\{(2,3,5),(3,5,8),(5,8,13)\}$?

Solution. Three elements in the three dimensional space form a basis if and only if they are linearly independent. The first set consist of linearly independent vectors, row reduction follows $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The second and the third set are not a basis because $(2,3,2) = (4,0,2)+(-2,3,0)$ and $(2,3,5)+(3,5,8) = (5,8,13)$, so the vectors in both sets are linearly dependent. $\square$

62. Find a basis and the dimension of the set of solutions of the system $\begin{cases} 2x + 6y + 4z = 2x, \\ -3x - 20y - 14z = 2y, \\ 6x + 35y + 24z = 2z. \end{cases}$

Solution. Move the right-hand sides to the left. We get a homogeneous system of linear equations with the matrix

$$\begin{pmatrix} 0 & 6 & 4 \\ -3 & -22 & -14 \\ 6 & 35 & 22 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ -3 & -22 & -14 \\ 0 & -9 & -6 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ -3 & -22 & -14 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ -3 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to$$

$\begin{pmatrix} 0 & 3 & 2 \\ -9 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. The system can be written now as $\begin{cases} 3y + 2z = 0, \\ -9x + 2z = 0. \end{cases}$ For any z there is a unique pair of x, y that satisfies the system. This proves that the set of solutions is a linear space of dimension 1. A basis we are looking for consists therefore of one element, e.g. $(2,-6,9)$. One can multiply this vector by any number $c \neq 0$ to obtain another basis consisting of one element $(2c,-6c,9c)$. $\square$

63. Find a basis and the dimension of the set of solutions of the system

$$\begin{cases} x_1 + 3x_2 + x_3 + 5x_4 = 2x_5 \\ 2x_1 + 7x_2 + 9x_3 + 2x_4 = 4x_5 \\ 4x_1 + 13x_2 + 11x_3 + 12x_4 = 8x_5 \end{cases}$$

Solution. The matrix of this homogeneous system is

$$\begin{pmatrix} 1 & 3 & 1 & 5 & -2 \\ 2 & 7 & 9 & 2 & -4 \\ 4 & 13 & 11 & 12 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 5 & -2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 1 & 7 & -8 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -20 & 29 & -2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The initial system of linear equations is equivalent to the system (of two not of three equations) $\begin{cases} x_1 - 20x_3 + 29x_4 - 2x_5 &= 0 \\ x_2 + 7x_3 - 8x_4 &= 0 \end{cases}$. This means that given any triple (x_3, x_4, x_5) we can find x_1 and x_2 such that the five numbers $(x_1, x_2, x_3, x_4, x_5)$ will be a solution to the initial system of equations. This proves that the set of solutions of the system of linear equations is a linear space of dimension three. One can find its basis e.g. by looking at the solutions $(20, -7, 1, 0, 0)$, $(-29, 8, 0, 1, 0)$, $(2, 0, 0, 0, 1)$. The starting point was the basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of the space $\mathbb{R}^3$ (we think of coordinates (x_3, x_4, x_5)). □

$$-2r_1 + r_2 = -4r_1 + r_3 \iff 2r_1 + r_2 - r_3 = (0, 0, 0)$$

64. Let P_n be the following set of the polynomials of degree n or less i.e. the functions of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ where $a_0, a_1, a_2, \ldots, a_n$ are given numbers. Which of the sets are basis of P_2:
$\{1, 1+x\}$, $\{1, 1+x, 2+x\}$, $\{1, 1+x, 2+x^2\}$, $\{1, x, (2+x)^2\}$, $\{10+x, 11+x, (1+x)^2\}$?
Solution. The set of polynomials of the form $c + bx + ax^2$ is a linear space of dimension 3. Therefore each basis has to consist of three elements of the space. Therefore the first set is not a basis of P_2. The second set consists of linearly dependent polynomials: $1 + (1+x) = 2+x$ so it is not a basis. Another proof of this statement is: there are no numbers c_1, c_2, c_3 such that $c_1 \cdot 1 + c_2 \cdot (1+x) + c_3(2+x) = x^2$ for all x. The set $\{1, 1+x, 2+x^2\}$ is a basis of P_2 because its elements are linearly independent. It so because if $c_1 + c_2(1+x) + c_3(2+x^2) = 0$ for all $x \in \mathbb{R}$ then $c_3 = 0$ otherwise there are at most two numbers x for which $c_1 + c_2(1+x) + c_3(2+x^2) = 0$. If $c_3 = 0$ then $c_1 + c_2 + c_2x = c_1 + c_2(1+x) = 0$. If $c_2 \neq 0$ then there is exactly one number x such that $c_1 + c_2(1+x) = 0$, namely $x = -\frac{c_1+c_2}{c_2}$. Therefore $c_2 = 0$. If $c_2 = c_3 = 0$ then also $c_1 = 0$ because $c_1 = c_1 + c_2(1+x) + c_3(2+x^2) = 0$ for all real x. In the same way one can prove that the remaining two sets are basis of P_2. □

65. Let V_3 be the set consisting of all sequences (a_n) such that for all $n \in \{0, 1, 2, \ldots\}$ the following equality holds:

$$a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$$

holds. Prove that v_3 is a linear space. What is its dimension? Try to find a basis consisting of sequences for which n^{th} term can be described with simple and short formula as a function of n only.
Solution. If a sequence $(a_0, a_1, a_2, a_3, \ldots)$ satisfies the equation

$$a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n \tag{65}$$

and $c \in \mathbb{R}$ the the sequence $(ca_0, ca_1, ca_2, ca_3, \dots)$ satisfies the equation (65), too. If the sequences $(a_0, a_1, a_2, a_3, \dots)$ and $(b_0, b_1, b_2, b_3, \dots)$ satisfy the equation (65) then their sum $(a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$ also satisfies this equation i.e.

$$a_{n+3} + b_{n+3} = 3(a_{n+2} + b_{n+2}) - 3(a_{n+1} + b_{n+1}) + (a_n + b_n).$$

Therefore V_3 is a linear space. Its dimension is 3 because the three terms of this sequence determine go fully. The sequences

$(1,\ 0,\ 0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28, \dots),$

$(0,\ 1,\ 0, -3, -8, -15, -24, -35, -48, -63, \dots)$ and

$(0,\ 0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36, \dots)$

satisfying equation (65) are linearly independent so they form a basis of V_3. The problem is that not everybody can see formulas for their n^{th} terms. We can ask whether or not there are arithmetical sequences satisfying (65). If so then

$$a_n = a_0 + nd \text{ so } a_0 + (n+3)d = 3(a_0 + (n+2)d) - 3(a_0 + (n+1)d) + a_0 + nd$$

and it is not hard to see that this equation holds for all pairs of numbers a_0, d.
If a geometric sequence with a ratio $q \neq 0$ satisfies (65) then $q^{n+3} = 3q^{n+2} - 3q^{n+1} + q^n$ so $q^3 = 3q^2 - 3q + 1$. This is equivalent to $0 = q^3 - 3q^2 + 3q - 1 = (q-1)^3$ so 1 is the only number that satisfies this equation. The only geometrical sequences that satisfy the condition (65) are constant sequences. This is also true when $q = 0$.
We need one more sequence to obtain a basis (we already have two $(1, 1, 1, \dots)$, i.e. $a_n = 1$, and $(0, 1, 2, 3, 4, 5, \dots,\ a_n = n)$. Now students should try with other simple formulas e.g. $a_n = n^2$ (sometimes it is a fifth guess or seventh but usually if one does not give up quickly she or he tries n^2). We have to check if (65) is fulfilled.

$$(n+3)^2 - \big(3(n+2)^2 - 3(n+1)^2 + n^2\big) = n^2 + 6n + 9 - \big(3(n+2+n+1) + n^2\big) = 0$$

so the equality holds for all n. The three sequences $(1, 1, 1, \dots)$, (n) and (n^2) are linearly independent so they form a basis. This means that if a sequence (a_n) satisfies the equation (65) the there exist numbers a, b, c such that $a_n = a + bn + cn^2$ for all n. □

A remark. Sometimes it is good to guess in order to solve a problem. Some students are afraid of guessing. Nobody knows why, at least I do not.

Write a solution of one of the problems 62, 63 and one of the problems 64, 65 and send it to me. You may use your computer or another electronic device to type it. Then convert it to a .pdf file. It you write with your pen on a paper and the take a picture of it convert the picture into a .pdf file and the mail to me. Give a reasonable name to the file which you will send to me (e.g. your last name or the first two or three letters of it followed by the first letter of your first name in there is another person with the same last name in the group 1 or the group 2.