A remainder: elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of a linear space are called linearly independent if and only if the unique choice of real numbers $t_1, t_2, \ldots, t_n$ such that $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots + t_n\mathbf{v}_n = \mathbf{0}$ is $t_1 = t_2 = \ldots = t_n = 0$.

The following elements of $\mathbb{R}^4$: $(1,0,0,0)$, $(1,2,3,4)$, $(1,5,6,7)$ are linearly independent because if $(0,0,0,0) = t_1(1,0,0,0) + t_2(1,2,3,4) + t_3(1,5,6,7)$ then $t_1 + t_2 + t_3 = 0$, $2t_2 + 5t_3 = 0$, $3t_2 + 6t_3 = 0$ and $4t_2 + 7t_3 = 0$. We see the system of four linear homogeneous equations with three unknowns. Its matrix is (it will be reduced right away)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \\ 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \\ 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously the corresponding system has exactly one solution, namely $t_1 = t_2 = t_3 = 0$.

The vectors $(1,2,3,4,5,5)$, $(5,2,15,4,25,1)$, $(1,-2,3,-4,5,-7)$ are linearly dependent because $3(1,2,3,4,5,5) - (5,2,15,4,25,1) + 2(1,-2,3,-4,5,-7) = (0,0,0,0,0)$

1. Are the vectors $(2,3)$, $(-5,7)$, $(13,17)$ linearly independent?

2. Are the vectors $(1,2,3)$, $(2,-5,7)$, $(1,13,17)$ linearly independent?

3. For what $n \in \mathbb{N}$ the polynomials 1, x, x^2,...,x^n are linearly independent in the space of all polynomials?

4. Are polynomials $x(x-1)$, $x(x-2)$, $(x-1)(x-2)$ linearly independent in the space of all polynomials?

5. Are the vectors $(1,1,1)$, $(0,2,5)$, $(0,3,6)$ and $(0,4,7)$ linearly independent?

Examples of vector spaces.

1. $\mathbb{R}$ — the set of all real numbers is a vector is a vector space: we can add the real numbers, the addition satisfies usual requirements for addition which are listed at the beginning of professor Kędzierski's Lecture 2. This is an one dimensional vector space, because any two distinct real numbers are linearly dependent (denote the numbers by a, b, they are distinct so at least one of them is not equal to 0, let $b \neq 0$ and $c_1 = 1$, $c_2 = -\frac{a}{b}$, then $c_1 a + c_2 b = a - \frac{a}{b} \cdot b = 0$ and obviously $1 = c_1 \neq 0$.

2. $\mathbb{R}^2$ the set of all pairs of the real numbers. $(x,y) + (u,v) = (x+u, y+v)$ it is a definition of the sum; $c(x,y) = (cx, cy)$ it is a definition of the product of a number c and a point. Again all usual properties of both operations are satisfied.

3. $\mathbb{R}^3$ is the set of all triples of the real numbers. The sum of the two triples is defined by the formula $(x,y,z) + (u,v,w) = (x+u, y+v, z+w)$, the product of a number c and a triple

(x, y, z) is defined as follows $f \cdot (x, y, z) = (cx, cy, cz)$. It is not hard to check that the addition is commutative, associative, that the triple $(0, 0, 0)$ has the property $(x, y, z) + (0, 0, 0) = (x, y, z)$ and that there is no other triple with this property and that $(x, y, z) + (-x, -y, -z) = (0, 0, 0)$. These are the properties of addition. The properties of the multiplication of numbers by the vectors are also obvious.

4. The previous examples can be generalized. We consider now the set $\mathbb{R}^n$ of all tuples of length $n \in \mathbb{N}$ as we did in the previous examples for $n = 1, 2, 3$. We define

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and

$$c \cdot (x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)\,.$$

We can check that all properties of addition and multiplication by numbers are fulfilled. We have defined n–dimensional space.

5. Let I be an interval (non-degenerate). Let R^I denotes a set of all functions defined on the interval I with real values. In the case $I = (0, 1)$ one of the elements of R^I is the function f defined by the formula $f(x) = \frac{1}{x(1-x)}$, another function that belongs to R^I is a function g defined by the formulas $g(x) = x$ for all rational numbers $x \in I$ and $g(x) = -1$ for all irrational $x \in I$. The set R^I contains infinitely many elements and it is not possible to list all of them. We define $f + g$ for $f, g \in R^I$ in the following way: $(f + g)(x) = f(x) + g(x)$. Obviously $f + g = g + f$ and $f + (g + h) = (f + g) + h$. The function Θ defined as $\Theta(x) = 0$ for all $x \in I$ plays the same role in the set R^I as the number 0 in $\mathbb{R}$ that is $t + \Theta = f$ for all $f \in R^I$. The product cf of a number c and a function $f \in R^I$ is defined by the formula $(cf)(x) = cf(x)$ for all $x \in I$. Again verification of the properties of the sum and of the product is easy.
Let $c \in KI$ be an arbitrary point. Let $t \in \mathbb{R}$ and $V_t = \{f \in R^I : \quad f(c) = t$. If $f, g \in V_t$ then $(f + g)(c) = 2t$ This proves that if $t \neq 0$ then V_t is NOT a linear space. It is easy to see thatV_0 is a linear space.
In this example we can consider any set A in place of the interval I (no property of intervals was used in the description of R^I. You may see that $R^{\{1,2,3\}} = R^3$ because it is very natural to think of triples of real numbers as of functions defined on the set $\{1, 2, 3\}$.

6. Let V be the set of all sequences (a_n) such that $a_{n+2} = a_{n+1} + a_n$ for $n = 1, 2, 3, \ldots$ and W the set of all sequences (a_n) such that $a_{n+2} = a_{n+1} + a_n + 7$ for $n = 1, 2, 3, \ldots$. The set W is NOT a linear space because if $(a_n), (b_n) \in W$ then $(a_n + b_n) \notin W$ for then $a_{n+2} + b_{n+2} = (a_{n+1} + b_{n+1}) + (a_n + b_n) + 2$. The set V is a linear space because if $(a_n), (b_n) \in V$ and $c \in \mathbb{R}$ then $(ca_n) \in V$ and $(a_n + b_n) \in V$: $ca_{n+2} = ca_{n+1} + ca_n$ and $a_{n+2} + b_{n+2} = (a_{n+1} + b_{n+1}) + (a_n + b_n)$

Let us solve the problem 3: For what $n \in \mathbb{N}$ the polynomials 1, x, x^2,. . . ,x^n are linearly independent in the space of all polynomials?

We shall prove that this happens for all $n = 1, 2, 3, \dots$ Let us discuss in detail the case $n = 2$. We are considering three functions $1, x, x^2$ defined on the whole real line. The question is whether or not it follows that from the equation $c_1 + \cdot 1 + c_2 + \cdot x + c_2 \cdot x^2 = 0$ for all $x \in \mathbb{R}$ that $c_1 = c_2 = c_3 = 0$. The equation $c_1 + \cdot 1 + c_2 + \cdot x + c_2 \cdot x^2 = 0$ holds for all $x \in \mathbb{R}$ because the functions $1, x, x^2$ are considered as defined on $\mathbb{R}$. Therefore we can substitute any real number for x. Let us start with $x = 0$. Then $c_1 + c_2 \cdot 0 + c_2 \cdot 0^2 = 0$ so $c_1 = 0$. We may also consider the equations $c_1 + c_2 \cdot 1 + c_2 \cdot 1^2 = 0$ and $c_1 + c_2 \cdot (-1) + c_2 \cdot (-1)^2 = 0$. This implies that $c_2 + c_3 = 0$ and $-c_2 + c_3 = 0$. If we add the last two equations we get $2c_3 = 0$ so $c_3 = 0$. Therefore $c_2 = 0$, too. This proves that the functions (polynomials) $1, x, x^2$ are linearly independent.

Now more general and more advanced approach. Suppose that there is some natural number n such that for each $x \in \mathbb{R}$ the equation

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 + \ldots + c_n \cdot x^n = 0$$

is satisfied. With n and $c_0, c_1, c_2, c_3, \dots, c_n$ the left–hand side is a polynomial in x. We can assume that $c_n \neq 0$ because we can forget of the terms with $c_j = 0$. If so then the degree of the polynomial is n and it there exist at most n numbers x for which the left–hand side assumes value 0 (n^{th} degree polynomial has at most n roots. $c_0 = c_1 = c_2 = c_3 = \ldots = c_n = 0$. This means that the polynomials $1, x, x^3, x^3, \dots, x^n$ are linearly independent for any value of n.

Let us discuss problem 4 now: Are polynomials $x(x-1)$, $x(x-2)$, $(x-1)(x-2)$ linearly independent in the space of all polynomials?

Suppose that there for some numbers c_1, c_2, c_3 the following equality:

$$c_1 x(x-1) + c_2 x(x-2) + c_3 (x-1)(x-2) = 0 \tag{4}$$

holds for each $x \in \mathbb{R}$. Substitute 0 for x. The result is $2c_3 = 0$, so $c_3 = 0$. Now let $x = 1$. We see that $-c_2 = 0$ so $c_2 = 0$. The last substitution is $x = 2$ with the result $c_1 - 0$. So the only triple c_1, c_2, c_3 that satisfies the condition (4) is $c_1 = c_2 = c_3 = 0$. The polynomials are linearly independent.

The polynomials of this sort appeared in the solution of a problem in the file *More systems of linear equations.* In the problem appeared x_1, x_2, x_3 rather than $0, 1, 2$. One can think about the situation in the following way. The set of the polynomials of degree 2 or less is a linear space of dimension 3 (such a polynomial can be written as $c + bx + ax^2$). Standard basis in this space consists of the polynomials $1, x, x^2$. More suitable for the problem discussed then was a basis $(x - x_2)(x - x_3), (x - x_1)(x - x_3), (x - x_1)(x - x_2)$. So we ended in the solution with this basis. A slightly different way of thinking is. We have a usual system of coordinates in this three dimensional space. The axes are determined by the vectors (polynomials) $1, x, x^2$. We choose another system of coordinates. The new axes are determined by the polynomials $(x - x_2)(x - x_3)$, $(x - x_1)(x - x_3)$, $(x - x_1)(x - x_2)$. The new coordinates work better for the solution.

Another part of the story is Fibonacci numbers. we were looking at the set of all sequences (a_n)

for which the equation

$$a_{n+2} = a_{n+1} + a_n \tag{fib}$$

holds for $n = 1, 2, 3, \ldots$ The set of these sequences turns out to be two dimensional space. The sequence whose terms atisfy the condition (fib) is determined by its first two terms. It turned out that for finding a formula for the n^{th} term of the Fibonacci sequence the better basis consisted of $\big(1, \frac{1}{2}(1-\sqrt{5})\big)$ and $\big(1, \frac{1}{2}(1+\sqrt{5})\big)$. The sequences starting with these pairs of terms satisfying (fib) were geometrical so there was no problem with writing a formula for their n^{th} terms.