Consider the Fibonacci sequence again : 1, 1, 2, 3, 5, 8, 13, 21, 34, ... A next term is a sum of its two predecessors: $F_{n+2} = F_{n+1} + F_n$. We may look at it in the following way. Out of a pair of given numbers x, y we create a new pair y and $x + y$ starting from $1, 1$. We may say that there is a map of $\mathbb{R}^2$ into itself given by the formula $f(x, y) = (y, x + y)$. The map is linear. We may write $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix} \to \begin{pmatrix} 3 \\ 5 \end{pmatrix} \to \begin{pmatrix} 5 \\ 8 \end{pmatrix}$ and so on. We can see that $\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This means that finding a formula for F_n is equivalent to finding a formula for $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$. Looking at powers of the matrix is not very helpful $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$. We see the Fibonacci numbers again which is not surprising at all becausde we only restated the problem. What we see is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n =$. Now we shall try to change the basis in $\mathbb{R}^2$ in an appropriate way. Let us try to find vectors mapped the vectors parallel to the that is such a non-zero vector $\begin{pmatrix} x \\ y \end{pmatrix}$ that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = q\begin{pmatrix} x \\ y \end{pmatrix}$ for some number q. Two equations must be fulfilled $y = qx$ and $x + y = qy$. This implies that $x + qx = q^2x$. If $x \neq 0$ then $1 + q = q^2$ so $q = \frac{1}{2}(1 \pm \sqrt{5})$. Let $\mathbf{v}_- = \begin{pmatrix} 1 \\ \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$ and $\mathbf{v}_+ = \begin{pmatrix} 1 \\ \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$, also $q_- = \frac{1}{2}(1 - \sqrt{5})$ and $q_+ = \frac{1}{2}(1 + \sqrt{5})$. We have $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ q_\pm \end{pmatrix} = q_\pm\begin{pmatrix} 1 \\ q_\pm \end{pmatrix}$. The vectors $\begin{pmatrix} 1 \\ q_- \end{pmatrix}, \begin{pmatrix} 1 \\ q_+ \end{pmatrix}$ are linearly independent so they constitute a basis of $\mathbb{R}^2$. The matrix of the map is $M_B^B = \begin{pmatrix} q_- & 0 \\ 0 & q_+ \end{pmatrix}$.

Obviously $(M_B^B)^n = \begin{pmatrix} q_-^n & 0 \\ 0 & q_+^n \end{pmatrix}$.

If $\mathbf{v} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ q_- \end{pmatrix} + u_2\begin{pmatrix} 1 \\ q_+ \end{pmatrix}$. This means that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q_- & q_+ \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =$ $= \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ or $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Using the notation adopted in the prof. Kędzierski's class we may write $M_{st}^B = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ and $M_B^{st} = \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}$. Można więc napisać, że $\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} q_-^n & 0 \\ 0 & q_+^n \end{pmatrix}\begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} =$

$= \begin{pmatrix} q_-^n & q_+^n \\ q_-^{n+1} & q_+^{n+1} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} q_+^{n-1} - q_-^{n-1} & q_+^n - q_-^n \\ q_+^n - q_-^n & q_+^{n+1} - q_-^{n+1} \end{pmatrix}$. It follows from the above equalities that

$$F_n = \frac{1}{\sqrt{5}}\left(q_+^n - q_-^n\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The change of the basis made the problem trivial (very easy). The whole story was told before without mentioning any basis, no matrices were multiplied. Now it was put into more general setting. One may imagine that similar actions may be undertaken in other situations.

90. What are the coordinates of the vector $\mathbf{v}$ relative to a basis B if
a. $\mathbf{v} = \binom{2}{3}$ and $B = \left\{\binom{1}{1}, \binom{1}{-1}\right\}$ **b.** $\mathbf{v} = \binom{3}{3}$ and $B = \left\{\binom{1}{2}, \binom{1}{1}\right\}$
c. $\mathbf{v} = \binom{12}{3}$ and $B = \left\{\binom{0}{2}, \binom{-1}{1}\right\}$ **d.** $\mathbf{v} = \binom{0}{9}$ and $B = \left\{\binom{10}{1}, \binom{-12}{1}\right\}$

91. Find a basis B of $\mathbb{R}^2$ such that the map f defined as $f(x, y) = (2x + 5y, 3x + 8y)$ relative to standard coordinates is represented by a matrix with zeroes outside of the main diagonal when the coordinates relative to B are used (the basis B is used for both the domain and the range) .

Solution. We have $M_{st}^{st}(f) = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. We want to find a basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ such that $M_B^B(f) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ for some numbers $c, d \in \mathbb{R}$. If the numbers c, d end the vectors $\mathbf{v}_1, \mathbf{v}_2$ exist then $f(\mathbf{v}_1) = c\mathbf{v}_1$ and $f(\mathbf{v}_2) = d\mathbf{v}_2$. The vectors $\mathbf{v}_1, \mathbf{v}_2$ should be linearly independent so none of them could be $(0, 0)$. Let $\mathbf{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ then $c\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 5y \\ 3x + 8y \end{pmatrix}$. This means that the equations $(2-c)x+5y = 0$ and $3x+(8-c)y = 0$ should hold. Subtract the first one multiplied by $8 - c$ from the second one multiplied by 5. Then $(15 - (2 - c)(8 - c))x = 0$. Since $x \neq 0$ we have $15 - (2 - c)(8 - c) = 0$ so $0 = c^2 - 10c + 1 = (c - 5)^2 - 24$ therefore $c = 5 \pm 2\sqrt{6}$. Let $c = 5+2\sqrt{6}$ and $x = 5$. Then $0 = (2-5-2\sqrt{6})5+5y = (-3-2\sqrt{6})5+5y$ so $y = 3+2\sqrt{6}$. Let $d = 5-2\sqrt{6}$ and $x = 5$. Then $0 = (2-5+2\sqrt{6})5+5y = (-3+2\sqrt{6})5+5y$ so $y = 3 - 2\sqrt{6}$. Then $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 + 2\sqrt{6} \end{pmatrix} = \begin{pmatrix} 25 + 10\sqrt{6} \\ 39 + 16\sqrt{6} \end{pmatrix} = (5 + 2\sqrt{6})\begin{pmatrix} 5 \\ 3 + 2\sqrt{6} \end{pmatrix}$ and $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 - 2\sqrt{6} \end{pmatrix} = \begin{pmatrix} 25 - 10\sqrt{6} \\ 39 - 16\sqrt{6} \end{pmatrix} = (5 - 2\sqrt{6})\begin{pmatrix} 5 \\ 3 - 2\sqrt{6} \end{pmatrix}$. Now define $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 + 2\sqrt{6} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 3 - 2\sqrt{6} \end{pmatrix}$. The above equalities prove that $M_B^B(f) = \begin{pmatrix} 5 + 2\sqrt{6} & 0 \\ 0 & 5 - 2\sqrt{6} \end{pmatrix}$ as required. □

Remark. In fact the 2 long equations at the end which contain matrices are not necessary because they follow immediately from what was said before.

92. Is the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ linear if $f(x,y) =$

a. $((x+3)^2-(x+1)^2-8, y)$;

b. $((x+3)^2-(x+1)^2, y)$;

c. $(2\sqrt[3]{x^3+3x^2+3x+1}-2(y+1), 5x-3y)$;

d. $(|x+1|-|y+1|, 2x)$.

93. Find a formula for the linear map $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ if $\varphi(2,3)=(1,0,1)$ and $\varphi(1,2)=(2,3,3)$.

94. Find a formula for the symmetry of $\mathbb{R}^3$ relative to the plane spanned by the vectors $(3,2,1)$ and $(1,-3,3)$. *A vector* (a,b,c) *is perpendicular to a vector* (u,v,w) *if and only if* $au+bv+cw=0$.

Solution. Let us start with finding a non–zero vector (a,b,c) perpendicular to the vectors $(3,2,1)$ and $(1,-3,3)$. The two equations should be satisfied: $3a+2b+c=$ and $a-3b+3c=0$. The matrix of this system of the linear homogeneous equations is
$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -8 \\ 1 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -8 \\ 11 & 0 & 9 \end{pmatrix}$. This means that $11b-8c=0$ and $11a+9c=0$ so we can set $c=11$, $b=8$ and $a=-9$. There are infinitely many other possibilities but they they do not differ too much from our choice: one can multiply the chosen vector by any number different from 0. The symmetry leaves all vectors in the plane spanned by the 2 vectors unchanged so $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is mapped to itself and also the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ is mapped to itself. The vector $\mathbf{v}_3 = \begin{pmatrix} -9 \\ 8 \\ 11 \end{pmatrix}$ is mapped to $-\mathbf{v}_3 = \begin{pmatrix} 9 \\ -8 \\ -11 \end{pmatrix}$.
This implies that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is mapped to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - c_3\mathbf{v}_3$. This means that the matric of the symmetry relative to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We want to find the matrix of the symmetry relative to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ etc. If
$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ i.e $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ then
$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. It remains to find $M^{-1} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}^{-1}$. The columns of $M = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}$ are mutually perpendicular so

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \\ -9 & 8 & 11 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 266 \end{pmatrix}$$. Therefore we may write $M^{-1} = \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix}$ so $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. We are ready to write an answer next to the final. We change coordinates from $x's$ to $c's$ then apply the symmetry and then we go back to $x's$ (read it from right to left):

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \\ -9 & 8 & 11 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$
$$= \begin{pmatrix} 3 & 2 & -1 \\ 1 & -3 & -3 \\ -9 & 8 & -11 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{52}{133} & \frac{72}{133} & \frac{99}{133} \\ \frac{72}{133} & \frac{69}{133} & \frac{-88}{133} \\ \frac{99}{133} & \frac{-88}{133} & \frac{12}{133} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

95. Let $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map such that $\varphi(3,1) = (1,2,3)$ and $\varphi(5,2) = (2,1,3)$ and $\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a symmetry relative to the plane spanned by the vectors $(3,2,1$ and $(1,-3,3)$. Find the matrix of the map $\psi \circ \varphi$ relative to the standard bases.

Solution. Let us find the matrix of the of φ. The matrix should have two columns and three rows because φ maps $\mathbb{R}^2$ into $\mathbb{R}^3$. Let it be $\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. We want to find such numbers a, b, c, d, e, f that $\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$. We have $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ this follows from the definition of the inverse matrix. Do not be afraid of guessing! Now multiply the equation by $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$. We get

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix}.$$

We may write $\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 3x - 7y \\ 3x - 6y \end{pmatrix}$ and we are done. The last thing is to find the matrix of $\psi \circ \varphi$. The matrix cam be obtained as a product of the two matrices $\begin{pmatrix} \frac{52}{133} & \frac{72}{133} & \frac{99}{133} \\ \frac{72}{133} & \frac{69}{133} & \frac{-88}{133} \\ \frac{99}{133} & \frac{-88}{133} & \frac{12}{133} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} = \frac{1}{133} \begin{pmatrix} 52 & 72 & 99 \\ 72 & 69 & -88 \\ 99 & -88 & 12 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} =$

$$= \frac{1}{133}\begin{pmatrix} 513 & -1046 \\ -57 & 117 \\ -228 & 643 \end{pmatrix}$$

96. Evaluate the determinants of the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}.$$

97. Evaluate the determinants of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} e^t & te^t & t^2e^t \\ e^t & (1+t)e^t & (2t+t^2)e^t \\ e^t & (2+t)e^t & (2+4t+t^2)e^t \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi\cos\theta & -r\cos\varphi\sin\theta & -r\sin\varphi\cos\theta \\ \cos\varphi\sin\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi & 0 & r\cos\varphi \end{pmatrix}.$$

Let us consider a system of the linear equations $\begin{cases} x_2 + 2x_3 + 3x_4 & = & 20 \\ x_1 + x_2 - x_3 + x_4 & = & 4 \\ 2x_1 - x_2 + x_3 + x_4 & = & 7 \\ 3x_1 + x_2 - x_3 - x_4 & = & -2 \end{cases}$. One can write formulas for x_1, x_2, x_3, x_4:

$(x_1, x_2, x_3, x_4) =$

$$= \left(\frac{\begin{vmatrix} 20 & 1 & 2 & 3 \\ 4 & 1 & -1 & 1 \\ 7 & -1 & 1 & 1 \\ -2 & 1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{vmatrix}}, \frac{\begin{vmatrix} 0 & 20 & 2 & 3 \\ 1 & 4 & -1 & 1 \\ 2 & 7 & 1 & 1 \\ 3 & -2 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{vmatrix}}, \frac{\begin{vmatrix} 0 & 1 & 20 & 3 \\ 1 & 1 & 4 & 1 \\ 2 & -1 & 7 & 1 \\ 3 & 1 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{vmatrix}}, \frac{\begin{vmatrix} 0 & 1 & 2 & 20 \\ 1 & 1 & -1 & 4 \\ 2 & -1 & 1 & 7 \\ 3 & 1 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{vmatrix}} \right).$$

This formula can be used succesfully for sytems of linear equations if the numbers of unknowns and equations are relatively small. In the case of huge numbers of unknowns and equations the amount of time that must be used for computing the determinants becames so big that even with big computers one has to wait „for ever”. Much better is Gauss elimination, one gets the results at a shorter time.

The system can be written in a matrix form:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 4 \\ 7 \\ -2 \end{pmatrix}$$

Solving the system is equivalent to finding the inverse matrix. Lets us start the work with few explanations. If we multiply the matrix $\begin{pmatrix} a & 0 & 0 & 0 \end{pmatrix}$ by the matrix $M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix}$ we get one row matrix $\begin{pmatrix} 0 & a & 2a & 3a \end{pmatrix}$ that is the matrix which consist of the first row of M multiplied by a. If we multiply the matrix $\begin{pmatrix} 0 & a & 0 & 0 \end{pmatrix}$ by M we get $\begin{pmatrix} a & a & -a & a \end{pmatrix}$ that is the matrix which consist of the second row of M multiplied by a. When we multiply the matrix $\begin{pmatrix} a & 0 & 1 & 0 \end{pmatrix}$ by M then we obtain one row matrix and this row is the sum of the third row of M and the first row of it multiplied by a. This implies that the product

$$\begin{pmatrix} 0 & 1 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1+3a & 1+a & -1-a & 1-a \\ 0 & 1 & 2 & 3 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix}$$

consists of the row which is the sum of row 2 and row 4 multiplied by a of the second factor, the first row of the second factor, and the third and the fourth rows of the second factor. The product $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & -1 & -1 \\ 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ consists of row 1 of the second factor, row 4 of the second factor, row 3 of the second factor and row 2 of the second factor. This shows how to interpret operations on rows of the matrix as multiplication from the left by an appropriate matrix. Let us show now how this allows to find the inverse of the matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix}.$$

We shall act on the rows of the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which consists of the given matrix M followed by the unit 4×4 matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

— we swapped rows one and two which means that the big matrix was multiplied from the left by the matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 2 & -4 & 0 & -3 & 0 & 1 \end{pmatrix}$$

— row 1 times -2 was added to row 3 and multiplied by -3 was added to row 4, multiplication by the matrix:

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9 & 8 & 3 & -2 & 1 & 0 \\ 0 & 0 & 6 & 2 & 2 & -3 & 0 & 1 \end{pmatrix}$$

— add row two times 3 to row 3 and row two times 2 to row 4, multiplication by the matrix:

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9 & 8 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -\frac{10}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 1 \end{pmatrix}$$

—add row three multiplied by $-\frac{2}{3}$ to row four, multiplication by the matrix

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & 1 \end{pmatrix};$$

Until now the absolute value of the determinant derminant of the matrix made up of the first four columns remained unchanged, the sign of the determinant was changed once when the two rows were swapped. It will change now.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{8}{9} & \frac{1}{3} & -\frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix}$$

— multiply row 3 by $\frac{1}{9}$, row 4 by $-\frac{3}{10}$, multiplication by the matrix:

$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & -\frac{3}{10} \end{pmatrix};$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} & \frac{3}{10} \\ 0 & 1 & 2 & 0 & 1 & -\frac{3}{2} & -\frac{3}{5} & \frac{9}{10} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix}$$

— add row 4 multiplied by -1 to row 1, multiplied by -3 to row 2, multiplied by $-\frac{8}{9}$ to row 3, multiplication by the matrix:

$$A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & -\frac{3}{10} \end{pmatrix};$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{4}{15} & \frac{17}{30} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{7}{15} & \frac{11}{30} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix}$$

— add row 3 to row 1, row 3 multiplied by -2 to row 2, multiplication by the matrix:

$$A_7 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{7}{15} & \frac{11}{30} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix}$$ — subtract row 2 from row 1, multiplication by the matrix $A_8 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

This proves that:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{7}{15} & \frac{11}{30} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{15} & \frac{4}{15} \\ 0 & \frac{1}{2} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix} = \frac{1}{30} \begin{pmatrix} 0 & 0 & 6 & 6 \\ 10 & -5 & -14 & 11 \\ 10 & -20 & -2 & 8 \\ 0 & 15 & 6 & -9 \end{pmatrix}.$$

We may write that $M^{-1} = A_8 \cdot A_7 \cdot A_6 \cdot A_5 \cdot A_4 \cdot A_3 \cdot A_2 \cdot A_1$

As one can see to find the inverse matrix it is enough to perform operations according to the algorithm. One has to be patient because this work is not fascinating at all.

Let us say that for 2×2 matrices all theories are in fact unnecessary. If one knows the definition of the inverse matrix then he or she can find it without any clever theories. If one wants to multiply the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by a matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ to obtain as a product $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $ax + bz = 0$ and $cw + dy = 0$ and this suggests immediately that $x = tb$ and $z = -ta$ for some number t and $w = sd$ and $y = -sc$ for some number s. Then we get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} sd & tb \\ -sc & -ta \end{pmatrix} = \begin{pmatrix} s(ad-bc) & 0 \\ 0 & t(bc-ad) \end{pmatrix}$. Obviously if $ad - bc = 0$ then the problem has no solution at all. If $ad - bc \neq 0$ then we can write $s = \frac{1}{ad-bc}$ and $t = \frac{1}{bc-ad}$ and we are done. One needs the theories for matrices of a bigger size.

Problem 26 of 60 problems of ALWNEcw.pdf

Let a linear map $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by the formula $\varphi((x_1, x_2, x_3)) = (x_1 - x_2 + 4x_3, -3x_1 + 8x_3)$. Let $A = \{(3,4,1), (2,3,1), (5,1,1)\}, B = \{(3,1), (2,1)\}$. Find $M(\varphi)_A^B$ and $M(\varphi)_{st}^{st}$ (matrices of φ relative to the bases A, B and to the standard bases, respectively).

Solution. All vectors wil be written vertically. We have

$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This means that the matrix $M_A^{st} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ allows us to compute the coordinates of a vector relative to the standard basis provided its coordinates relative to the basis A are known. As we know the matrix M_{st}^A that allows to compute the coordinates of a vector relative to A when its coordinates relative to the standard basis are known is simply the inverse of M_A^{st} i.e. $M_{st}^A = (M_A^{st})^{-1}$. Let us find tha matrix $(M_A^{st})^{-1}$ with method advertised above:

$$\begin{pmatrix} 3 & 2 & 5 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 2 & 5 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 0 & -3 \\ 0 & -1 & -3 & 0 & 1 & -4 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 1 & -2 & -1 & 0 & 3 \\ 0 & 0 & -5 & -1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 1 & -2 & -1 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} & \frac{3}{5} & -\frac{13}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & -\frac{2}{5} & \frac{17}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$ This

proves that $M_{st}^A = (M_A^{st})^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & -\frac{13}{5} \\ -\frac{3}{5} & -\frac{2}{5} & \frac{17}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 3 & -13 \\ -3 & -2 & 17 \\ 1 & -1 & 1 \end{pmatrix}$.

We also know that $M_B^{st} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ so $M_{st}^B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$.

Now $\varphi \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-4+4 \\ -3\cdot 3+8 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\varphi \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-3+4 \\ -3\cdot 2+8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ oraz $\varphi \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} =$ $= \begin{pmatrix} 5-1+4 \\ -3\cdot 5+8 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \end{pmatrix}$. We want to switch to to coordinates relative to the basis B:

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ -7 \end{pmatrix} = \begin{pmatrix} 22 \\ -29 \end{pmatrix}.$$

This proves that $M_A^B(\varphi) = \begin{pmatrix} 5 & -1 & 22 \\ -6 & 3 & -29 \end{pmatrix}$. Therefore $M_{st}^{st}(\varphi) = M_B^{st} M_A^B(\varphi) M_{st}^A =$

$$= \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 & 22 \\ -6 & 3 & -29 \end{pmatrix} \begin{pmatrix} 2 & 3 & -13 \\ -3 & -2 & 17 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & 3 & 8 \\ -1 & 2 & -7 \end{pmatrix} \begin{pmatrix} 2 & 3 & -13 \\ -3 & -2 & 17 \\ 1 & -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{5} \begin{pmatrix} 5 & -5 & 20 \\ -15 & 0 & 40 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 4 \\ -3 & 0 & 8 \end{pmatrix}.$$ We are done. □

Remark. The above problem can be solved in a slightly different way. On does not have to use the matrices all the time. If we want to find the matrix of φ with respect to the basis A (of the domain of φ) and the basis B (of the range of φ) then we might have asked how to write the vectors $\varphi \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\varphi \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\varphi \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$ can be written as linear combinations of the elements of the set B i.e. of the vectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This leads to a system of the linear equations $3c_1 + 2c_2 = 3$ and $c_1 + c_2 = -1$ which can be solved without any matrices quickly: $c_1 = 5$, $c_2 = -6$.

The we do the same with the vector $\varphi\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}=c_1\begin{pmatrix}3\\1\end{pmatrix}+c_2\begin{pmatrix}2\\1\end{pmatrix}$. This implies that $c_1=-1$ and $c_2=3$. The last equation is $\varphi\begin{pmatrix}5\\1\\1\end{pmatrix}=\begin{pmatrix}8\\-7\end{pmatrix}=c_1\begin{pmatrix}3\\1\end{pmatrix}+c_2\begin{pmatrix}2\\1\end{pmatrix}$. Again there is no problem in getting $c_1=22$ and $c_2=-29$. We proved that $\varphi\begin{pmatrix}3\\4\\1\end{pmatrix}=5\begin{pmatrix}3\\1\end{pmatrix}-6\begin{pmatrix}2\\1\end{pmatrix}$.

This means that the first column of the matrix $M_A^B(\varphi)$ is $\begin{pmatrix}5\\-6\end{pmatrix}$. $\varphi\begin{pmatrix}2\\3\\1\end{pmatrix}=-\begin{pmatrix}3\\1\end{pmatrix}+3\begin{pmatrix}2\\1\end{pmatrix}$.

Therefore the second column of the matrix $M_A^B(\varphi)$ is $\begin{pmatrix}-1\\3\end{pmatrix}$. The last step in these considarations is $\varphi\begin{pmatrix}5\\1\\1\end{pmatrix}=\begin{pmatrix}8\\-7\end{pmatrix}=22\begin{pmatrix}3\\1\end{pmatrix}-29\begin{pmatrix}2\\1\end{pmatrix}$ and this proves that the third column of the matrix is $\begin{pmatrix}22\\-29\end{pmatrix}$. So the matrix $M_A^B(\varphi)$ is found. In a similar way we can find the matrix $M_{st}^{st}(\varphi)$. We shall not do it here because we did it above in a different way. In fact ther difference is mainly in the use of different words.

Problem 27 of 60 problems of ALWNEcw.pdf

Let $\mathcal{A}=\{(2,1),(1,1)\},\ \mathcal{B}=\{(1,3),(0,1)\},\ \mathcal{C}=\{(0,1),(1,4)\}$. and let $\varphi\colon\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be a linear map such that $M_{\mathcal{A}}^{\mathcal{B}}(\varphi)=\begin{pmatrix}1&2\\3&4\end{pmatrix}$. Find $M_{\mathcal{A}}^{\mathcal{C}}(\varphi)$.

Solution. We need to switch from the basis $\mathcal{B}$ to the basis $\mathcal{C}$. We have $M_{\mathcal{B}}^{st}=\begin{pmatrix}1&0\\3&1\end{pmatrix}$ therefore $M_{st}^{\mathcal{B}}=\begin{pmatrix}1&0\\-3&1\end{pmatrix}$, $M_{\mathcal{C}}^{st}=\begin{pmatrix}0&1\\1&4\end{pmatrix}$ therefore $M_{st}^{\mathcal{C}}=\begin{pmatrix}-4&1\\1&0\end{pmatrix}$. We may write

$$M_{\mathcal{B}}^{\mathcal{C}}=M_{st}^{\mathcal{C}}M_{\mathcal{B}}^{st}=\begin{pmatrix}-4&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\3&1\end{pmatrix}=\begin{pmatrix}-1&1\\1&0\end{pmatrix}\text{ and}$$

$$M_{\mathcal{A}}^{\mathcal{C}}(\varphi)=M_{\mathcal{B}}^{\mathcal{C}}M_{\mathcal{A}}^{\mathcal{B}}(\varphi)=\begin{pmatrix}-1&1\\1&0\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}2&2\\1&2\end{pmatrix}.$$

We are done. □

One can do it in a slightly different way. To find the matrix $M_{\mathcal{A}}^{\mathcal{C}}(\varphi)$ one has to know the images of the elements of $\mathcal{A}$ which we know but they are written as linear combinations of the basis $\mathcal{B}$ and we need to write them as linear combinations od the lements of the basis $\mathcal{C}$. This means that we should find numbers c_1, c_2, d_1, d_2 such that

$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \varphi\begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and

$\begin{pmatrix} 2 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \varphi\begin{pmatrix} 1 \\ 1 \end{pmatrix} = d_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + d_2\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Frtom these equations we get right away the equalities $c_2 = 1$, $c_1 = 6 - 4 \cdot c_2 = 2$, $d_2 = 2$ and $d_1 = 10 - 4 \cdot d_2 = 2$. Therefore $M_{\mathcal{A}}^{\mathcal{C}}(\varphi) = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$. □

The second method in fact coincides with the first one. The difference is that matrices are used not so often as in the previous wording.

Problem 30 of 60 problems of ALWNEcw.pdf

Let $A = \{(1,2,3),(2,1,0),(4,5,0)\}$, $B = \{(2,1,2),(3,1,2),(2,1,3)\}$. Find a matrix $C \in M_{3\times3}(\mathbb{R})$, fulfilling the following condition. For a given vector $\alpha \in \mathbb{R}^3$: if the coordinates of α relative to the basis A are x_1, x_2, x_3 and the coordinates of α relative to the basis B are y_1, y_2, y_3 then

$$C \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Let us notice at first that $M_A^{st} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 0 & 0 \end{pmatrix}$ and $M_B^{st} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}$. We need to find the matrix $M_A^B = M_{st}^B \cdot M_A^{st} = (M_B^{st})^{-1} \cdot M_A^{st}$ so we have to find $(M_B^{st})^{-1} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}^{-1}$. Let us do it

$\begin{pmatrix} 2 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix} \longrightarrow$

$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 5 & -1 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}$. This shows that $(M_B^{st})^{-1} = M_{st}^B =$

$= \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 5 & -1 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. Therefore

$M_A^B = \begin{pmatrix} -1 & 5 & -1 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 21 \\ -3 & 0 & -6 \\ -1 & -2 & -10 \end{pmatrix}$. We are done. □

The obtained reasult means that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = 21 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 6 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - 10 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$
We could have solved three systems of the linear equations instead of finding the matrices and their inverses. We shall write the first system
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ or $\begin{cases} 2c_1 + 3c_2 + 2c_3 = 1 \\ c_1 + c_2 + c_3 = 2 \\ 2c_1 + 2c_2 + 3c_3 = 3 \end{cases}$. You may write and then solve two other systems to see that one can essentially avoid using matrices in the solution except for the last sentence where they must appear because the answer requiers the a matrix.

Problem 39 of 60 problems of ALWNEcw.pdf

For the endomorphism $\varphi \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\varphi(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2)$ and for three bases $A_1 = \{(4,1),(3,1)\}$, $A_2 = \{(2,3),(5,8)\}$, $A_3 = \{(4,2),(1,1)\}$ find matrices $A_i = M(\varphi)_{A_i}^{A_i}$ for $i = 1,2,3$ and matrices C_{ij} fulfilling $A_j = C_{ij}^{-1} A_i C_{ij}$ for $i, j = 1,2,3$.

Solution. Let us take care at first of A_1. We have $\varphi \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 18 \end{pmatrix}$ and $\varphi \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \end{pmatrix}$. Now we want to express the vectors $\begin{pmatrix} 16 \\ 18 \end{pmatrix}$ and $\begin{pmatrix} 13 \\ 13 \end{pmatrix}$ as linear combinations of the elements of A_1 i.e. of the vectors $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So we want to find the numbers c_1, c_2 such that the following equations are satisfied $\begin{pmatrix} 16 \\ 18 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. This implies that $c_1 = 16 - 3 \cdot 18 = -38$ and $c_2 = 18 - c_1 = 56$. Also $\begin{pmatrix} 13 \\ 13 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ implies that $c_1 = 13 - 3 \cdot 13 = -26$ and $c_2 = 13 - c_1 = 39$. This implies that $M(\varphi)_{A_1}^{A_1} = \begin{pmatrix} -38 & -26 \\ 56 & 39 \end{pmatrix}$.

Slightly different method. $M_{A_2}^{st} = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \Rightarrow M_{st}^{A_2} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}$ so $M(\varphi)_{A_2}^{A_2} = M_{st}^{A_2} M(\varphi)_{st}^{st} M_{A_2}^{st} =$
$$= \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 18 & 47 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 124 & 331 \\ -46 & -123 \end{pmatrix}.$$

$$M_{A_2}^{st} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow M_{st}^{A_3} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \text{ so } M(\varphi)_{A_3}^{A_3} = M_{st}^{A_3} M(\varphi)_{st}^{st} M_{A_3}^{st} =$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 20 & 7 \\ 16 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 12 & -1 \end{pmatrix}.$$

Problem 31 of 60 problems of ALWNEcw.pdf

A linear map $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is given by the formula $\varphi(x_1; x_2; x_3) = (3x_1 + 7x_2 + 4x_3; x_1 + 2x_2 + x_3)$. Find bases A of $\mathbb{R}^3$ and B of $\mathbb{R}^2$, such that $M(\varphi)_A^B = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Solution. We have $M(\varphi)_{st}^{st} = \begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \end{pmatrix}$. We want to find the matrices M_A^{st} and M_{st}^B such that the equation

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = M_{st}^B M(\varphi)_{st}^{st} M_A^{st} = M_{st}^B \begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \end{pmatrix} M_A^{st}$$

will be satisfied. M_{st}^B should be a 2×2 matrix, M_A^{st} should be a 3×3 matrix. So essentially speaking we have now six equations with $4 + 9 = 13$ unknowns. This suggests the problem has infinitely many solutions. Some solutions of the system of the linear equations will not give us solutions of the problem because the matrices M_{st}^B and M_A^{st} must be invertible. We shall show how to find some bases A, B having in mind that there are many more solutions to the problem. We shall not change the basis in the range at all so we set $M_{st}^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. so we want to find a matrix M_A^{st} such that $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = M_{st}^{st} M_A^{st} = \begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \end{pmatrix} M_A^{st}$. Let us look at the equation $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} M_A^{st}$ (a third row was added to both non-square matrices to make them square and invertible). This implies that $\begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_A^{st}$. We start with finding the inverse of $\begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. It may be done faster than below.

$$\begin{pmatrix} 3 & 7 & 4 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 7 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 1 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -2 & 7 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & \mathbf{0} & 0 & 1 \end{pmatrix}.$$

We proved that $\begin{pmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 7 & 1 \\ 1 & -3 & -1 \\ \mathbf{0} & 0 & 1 \end{pmatrix}$. Therefore $M_A^{st} = \begin{pmatrix} -2 & 7 & 1 \\ 1 & -3 & -1 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This means that one of infinitely many solutions is $A = \left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. $\square$

Remark. It is easy to check that for any numbers $a, b, c \in \mathbb{R}$ φ maps the vectors

$$\begin{pmatrix} a \\ 2-a \\ -3+a \end{pmatrix}, \begin{pmatrix} b \\ -2-b \\ 4+b \end{pmatrix}, \begin{pmatrix} c \\ -1-c \\ 2+c \end{pmatrix}$$

onto the vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ – just multiply the matrices. Somewhat harder but still easy is to check there are no other triples of vectors mapped to these in the plane. One also can compute the determinant $\begin{vmatrix} a & b & c \\ 2-a & -2-b & -1-c \\ a-3 & b+4 & c+2 \end{vmatrix} = 2c - b$. This shows the if $b \neq 2c$ then the set of the vectors $\begin{pmatrix} a \\ 2-a \\ a-3 \end{pmatrix}, \begin{pmatrix} b \\ -2-b \\ 4+b \end{pmatrix}, \begin{pmatrix} c \\ -1-c \\ 2+c \end{pmatrix}$ is a basis of $\mathbb{R}^3$. For $a = 3, b = -4, c = -1$ one obtains the set at the end of the above solution, for $a = 2, b = -4, c = -1$ the set that appeared at professor Kedzierski's lecture some time ago. These are not all solutions of the problem because we did not change the basis in the range of φ at all.