1. Let $H = \{(x,y)\colon\quad xy+x+y=0\}$ and $L = \{(x,y)\colon\quad x+2y+1=0\}$. Find the minimal value of $\|\mathbf{p}-\mathbf{q}\|$ for $p \in H$ and $\mathbf{q} \in L$.

2. Let $C = \{(x,y,z)\colon\quad xy+yz+zx=0\}$ and $\mathbb{P} = \{(x,y,z)\colon\quad x+y+z=3\}$. Find the maximal distance from points of $\mathbb{P}\cap C$ to the z–axis.

3. Let $A = \{(x,y,z)\colon\quad z=x^2+y^2\}$, $B = \{(x,y,z)\colon\quad z=2x+2y-9\}$. Find the minimal distance from points of A to points of B.

4. Let $f(w,x,y,z) = w^3+x^3+y^3+z^3$. Find $\sup_A f$ and $\inf_A f$ if
$A = \{(w,x,y,z)\colon\quad w^2+x^2-wx=1 \quad \text{and} \quad y^2+z^2-yz=1\}$

5. Find $\sup\{x^4+y^4\colon\quad x^2+y^2-xy=3\}$ and $\inf\{x^4+y^4\colon\quad x^2+y^2-xy=3\}$.

6. Let $B = \{(x,y,z)\colon\quad x^2+y^2 \leqslant z^2 \leqslant 3(x^2+y^2),\ 0 \leqslant z \leqslant 2\}$. Compute the integral

$$\iiint_B \sqrt{x^2+y^2+z^2}\,dxdydz.$$

7. Find the area of the set A which is contained in the first quadrant and bounded by the curves: $\quad y=x, \quad y=2x, \quad xy=3, \quad xy=4.$

8. Compute the volume of the set $A = \{(x,y,z)\in\mathbb{R}^3\colon\quad x^2+y^2 \leqslant 1,\quad x \geqslant |y| \geqslant |z|\}$.

9. Compute the volume of the set $A = \{(x,y,z)\colon\quad x>0,\ y>0,\ xy<z,\ x^4+z^4<x^2z\}$.

1. Let $H = \{(x,y)\colon \quad xy + x + y = 0\}$ and $L = \{(x,y)\colon \quad x + 2y + 1 = 0\}$. Find the minimal value of $\|\mathbf{p} - \mathbf{q}\|$ for $p \in H$ and $\mathbf{q} \in L$.

Solution. The first observation is $H \cap L = \emptyset$. If $(x,y) \in H \cap L$ then $x = -2y - 1$ and $0 = xy + x + y = y(-2y-1) - (2y+1) + y = -2y^2 - 2y - 1 = -y^2 - (y+1)^2 < 0$.

Let $\mathbf{p} = (w,x)$ and $\mathbf{q} = (y,z)$. We are supposed to find the minimal value the function $d(w,x,y,z) = (w-y)^2 + (x-z)^2$ (formally speaking $\sqrt{(w-y)^2 + (x-z)^2}$ but we can forget of the square root until the very end) subject to the constraints

$$wx + w + x = 0 \quad \text{and} \quad y + 2z + 1 = 0. \tag{1}$$

We have $\nabla(wx+w+x) = (x+1, w+1, 0, 0)$, $\nabla(y+2z+1) = (0,0,1,2)$ These vectors are linearly independent unless $w = -1 = x$ but in this case $(-1)(-1) + (-1) + (-1) \neq 0$. Therefore we can apply the Lagrange theorem. If d attains its extreme value at a point (w,x,y,z) then there exist numbers λ_1, λ_2 such that

$$\nabla((w-y)^2 + (x-z)^2) = \lambda_1 \nabla(wx + w + x) + \lambda_2 \nabla(y + 2z + 1)$$

which means that the following four equations are satisfied

$2(w-y) = \lambda_1(x+1), \qquad 2(x-z) = \lambda_1(w+1), \qquad 2(y-w) = \lambda_2, \qquad 2(z-x) = 2\lambda_2.$

If $\lambda_1 = 0$ or $\lambda_2 = 0$ then $(w,x) = (y,z)$, a contradiction. Therefore $\lambda_1 \neq 0 \neq \lambda_2$. The first two equations imply that (left times right equals right times left, then $\lambda_1 \neq 0$)

$$(w-y)(w+1) = (x-z)(x+1). \tag{2}$$

The third and the fourth one imply that

$$2(y-w) = z - x. \tag{3}$$

Thus $x = z$ if and only if $w = y$, so $x \neq z$ and $w \neq y$. From (2) and (3) we obtain $2(x+1) = w+1$ i.e. $w = 2x+1$ so $0 = x(2x+1) + 2x + 1 + x = 2x^2 + 4x + 1 = 2(x+1)^2 - 1$. This implies that either $x = -1 - \frac{1}{\sqrt{2}}$ or $x = -1 + \frac{1}{\sqrt{2}}$. The corresponding values of w are $-1-\sqrt{2}$ and $-1+\sqrt{2}$. We still need y and z. By (2) we get that $2y - z = 2w - x$. (1) implies $y + 2z = -1$. This implies that $5y = 2(2w-x) + (-1) = 4w - 2x - 1 = 3w$ so $y = \frac{3}{5}w$. Therefore $z = 2y - 2w + x = \frac{6}{5}w - 2w + \frac{w-1}{2} = -\frac{3}{10}w - \frac{1}{2}$. This implies that there are two possible quadruplets (w,x,y,z): $\big(-1-\sqrt{2},\ -1-\frac{1}{\sqrt{2}},\ -\frac{3}{5}(1+\sqrt{2}),\ \frac{1}{10}(-2+3\sqrt{2})\big)$ and $\big(-1+\sqrt{2},\ -1+\frac{1}{\sqrt{2}},\ \frac{3}{5}(-1+\sqrt{2}),\ -\frac{1}{10}(2+3\sqrt{2})\big)$. It remains to evaluate d at the points:

$d(-1-\sqrt{2},\ -1-\frac{1}{\sqrt{2}},\ -\frac{3}{5}(1+\sqrt{2}),\ \frac{1}{10}(-2+3\sqrt{2})) = \frac{4}{25}(1+\sqrt{2})^2 + \frac{16}{25}(1+\sqrt{2})^2 = \frac{4}{5}(3+2\sqrt{2})$,

$d(-1+\sqrt{2},\ -1+\frac{1}{\sqrt{2}},\ \frac{3}{5}(-1+\sqrt{2}),\ -\frac{1}{10}(2+3\sqrt{2})) = \frac{4}{25}(1-\sqrt{2})^2 + \frac{16}{25}(1-\sqrt{2})^2 =$
$=\frac{4}{5}(3-2\sqrt{2})$. We know now that **if** the smallest value of the function d exists then it equals to $\frac{4}{5}(3-2\sqrt{2})$. Both sets H and L are closed and both are unbounded. Therefore we cannot call upon Weierstrass min-max theorem right away.

Let us assume that $w^2 + x^2 \geqslant 2 \cdot 10^8$ and $y^2 + z^2 \geqslant 2 \cdot 10^8$. If $(w,x) \in H$ then $x = \frac{-w}{w+1}$ and $w = \frac{-x}{x+1}$. Either $w^2 \geqslant 10^8$ or $x^2 \geqslant 10^8$ for otherwise $w^2 + x^2 < 2 \cdot 10^8$ so either $|w| \geqslant 10^4$ or $|x| \geqslant 10^4$. Assume that $|w| \geqslant 10^4$. Then $|x| = \left|-1 + \frac{2}{w+1}\right| = 1 + \frac{2}{|w|-1} < 2$. If $(w-y)^2 + (x-z)^2 < 1$ then $|w| - |y| \leqslant |w-y| < 1$ so $|y| > |w| - 1 \geqslant 10^4 - 1$ and therefore by (1) one gets $|z| = |\frac{-1-y}{2}| \geqslant \frac{|y|-1}{2} > \frac{1}{2}(10^4 - 2)$. Then

$$1 > |x - z| \geqslant |z| - |x| > \tfrac{1}{2}(10^4 - 2) - 2 > 1000,$$

a contradiction. The same argument works for $|x| \geqslant 10^4$. So we may consider the set consisting of points for which $w^2+x^2 \leqslant 2\cdot 10^8$ and $y^2+z^2 \leqslant 2\cdot 10^8$ and of course $wx+w+x=0$ and $y+2z+1=0$. This set is bounded and closed, d is a continuous function so it has a minimal value and from what was written above it follows that the smallest value is attained at a point (w,x,y,z) for which $w^2+x^2 < 2\cdot 10^8$ and $y^2+z^2 < 2\cdot 10^8$ so at one of the two points found earlier. □

Remark 9.1 The idea of the proof that the minimal value exists id rather simple.The equation $wx+w+x=0$ is equivalent to $(w+1)(x+1)=1$. Therefore if $|w|$ is huge then $|x+1|\approx 0$ so $x\approx -1$. Then if $|w-y|$ is small the y is huge. But $|z|=\frac{|y+1|}{2}$ so $|z|\approx\frac{|y|}{2}$ so $|z|$ is huge and therefore it cannot be approximately equal to $x\approx -1$. □

6. Let $B=\{(x,y,z)\colon \quad x^2+y^2\leqslant z^2\leqslant 3(x^2+y^2),\ 0\leqslant z\leqslant 2\}$. Compute the integral

$$\iiint_B \sqrt{x^2+y^2+z^2}dxdydz.$$

Solution. x^2 is always accompanied by y^2. Thus it makes sense to substitute $x=r\cos\theta$ and $y=r\sin\theta$ with $0<r\leqslant z$ and $-\pi<\theta<\pi$. We have $\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$. The inequality $x^2+y^2<z^2$ is equivalent to $r<z$, the inequality $z^2<3(x^2+y^2)$ is equivalent to $\frac{z}{\sqrt{3}}<r$. This implies that (by the chain rule $\frac{\partial}{\partial r}\left((r^2+z^2)^{3/2}\right)=3r\sqrt{r^2+z^2}$)

$$\iiint_B \sqrt{x^2+y^2+z^2}dxdydz=\int_0^2\int_{z/\sqrt{3}}^{z}\int_{-\pi}^{\pi}\sqrt{r^2+z^2}rd\theta drdz=$$

$$=2\pi\int_0^2\left(\frac{1}{3}(r^2+z^2)^{3/2}\Big|_{z/\sqrt{3}}^{z}\right)dz=\frac{2\pi}{3}\int_0^2\left(2^{3/2}z^3-\left(\tfrac{4}{3}\right)^{3/2}z^3\right)dz=$$

$$=\tfrac{2\pi}{3}\left(2^{3/2}-\left(\tfrac{4}{3}\right)^{3/2}\right)\left(\tfrac{2^4}{4}-\tfrac{0^4}{4}\right)=\tfrac{8\pi}{3}\left(2^{3/2}-\left(\tfrac{4}{3}\right)^{3/2}\right)=\tfrac{16\pi}{3}\left(\sqrt{2}-\tfrac{4}{3\sqrt{3}}\right).\ \square$$

Remark 9.2 The order of integration was not random. It is easier to integrate $\int r\sqrt{r^2+z^2}dr$ than $\int r\sqrt{r^2+z^2}dz$. Also the result is simpler and this may simplify next integration. □