Lagrange multipliers

Let us look at the ninth problem from the April colloquium.

9. Let $A = \{(x, y)\colon \quad \frac{x^2}{4} + \frac{y^2}{9} \leqslant 1\}$ and let $f\colon A \longrightarrow \mathbb{R}$ be case (k_1) below

(1) $f(x,y) = 3x^3 + 2y^2$ (2) $f(x,y) = x^5 - 2y^2$ (3) $f(x,y) = x^2 + y^3$

(4) $f(x,y) = 27x^2 + 2y^3$ (5) $f(x,y) = 6x^2 + y^3$ (6) $f(x,y) = 45x^2 + 2y^5$.

What are the critical points of f?

What are the maximum and minimum values of f on the boundary of A?

What are the maximum and minimum values of f on A?

Each question is worth 1 point.

The set A is compact. All considered functions are continuous on A. By Weierstrass maximum principle each of them attains its least upper bound and greatest lower bound. The point at which an extreme value is attained may lie inside of the domain and then it must be a critical point of the function or it may be a boundary point not necessarily critical.

In all cases the only critical point is $(0,0)$ — straightforward calculation. The value of each function at the origin is 0.

It is neither the biggest value of the function nor the smallest one: in the first and in the second case we look at $f(x,0)$. This is $3x^3$ or x^5 so it is positive at many points and negative at many other points. At all other cases we look at the function $f(0,y)$ and as in the previous case it is odd degree monomial so its values are positive at many points and negative at many others. This means that in all cases maximal and minimal values are attained at some boundary points so the answers to the second and to the third questions coincide. We shall look at the boundary points only i.e.

$$g(x,y) := \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0. \tag{i}$$

We compute $\nabla g(x,y) = \left(\frac{x}{2}, \frac{2y}{9}\right) \neq (0,0)$ for $(x,y) \in A$

(1) $f(x,y) = 3x^3 + 2y^2$. By Lagrange theorem there exists λ such that

$$(9x^2, 4y) = \nabla f(x,y) = \lambda g(x,y) = \lambda\left(\tfrac{x}{2}, \tfrac{2y}{9}\right). \tag{ii}$$

Therefore

$$\lambda x = 18x^2 \quad \text{and} \quad \lambda y = 18y \tag{iii}$$

By the second equation of *(iii)* either $y = 0$ or $\lambda = 18$. If $y = 0$ then by *(i)* we have $x^2 = 4$ so $x = \pm 2$. If $y \neq 0$ then $\lambda = 18$ so $x = x^2$ hence $x = 0$ or $x = 1$. Therefore there are six possibilities $(2,0)$, $(-2,0)$, $(0,3)$, $(0,-3)$, $\left(1, \frac{\sqrt{27}}{4}\right)$ and $\left(1, -\frac{\sqrt{27}}{4}\right)$ for a point at which f attains its maximal or minimal value. The corresponding values are 24, -24, 18, 18, $\frac{1}{4} + \frac{27}{16}$ and again $\frac{1}{4} + \frac{27}{16}$. The biggest is 24. The smallest is -24. We are done. $\square$

(6) $f(x,y) = 45x^2 + y^5$. By Lagrange theorem there exists λ such that

$$(90x, 5y^4) = \nabla f(x,y) = \lambda g(x,y) = \lambda\left(\tfrac{x}{2}, \tfrac{2y}{9}\right). \tag{ii}$$

Therefore

$$\lambda x = 180x \quad \text{and} \quad 2\lambda y = 45y^4 \tag{iii}$$

By the first equation of (iii) either $x = 0$ or $\lambda = 180$. If $x = 0$ then by (i) we have $y^2 = 9$ so $y = \pm 3$. If $x \neq 0$ then $\lambda = 180$ so $360y = 45y^4$ hence $y = 0$ or $y^3 = 8$ i.e. $y = 2$. Therefore there are six candidates $(0, 3)$, $(0, -3)$, $(2, 0)$, $(-2, 0)$, $\left(\sqrt{\frac{20}{9}}, 2\right)$ and $\left(-\sqrt{\frac{20}{9}}, 2\right)$ for a point at which f attains its maximal or minimal value. The corresponding values are 243, -243, 180, 180, 132 and 132. This proves that $\max f = 243$ and $\min f = -243$. $\square$.

All other cases are similar so they will not be discussed again.

Now let us look into the example from professor Warhurst's notes.

Example 8.1 Let $f(x, y, z) = xy + z$ constrained to the unit sphere by

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0. \tag{1}$$

We shall fund the maximal and the minimal value of the function f on the sphere (1). Let us start with an observation $\nabla g(x, y, z) = (2x, 2y, 2z) \neq (0, 0, 0)$ for (x, y, z) satisfying (1). This allows to apply Lagrange multipliers. The set M defined by (1) is closed and bounded so it is compact. The function f is continuous everywhere so it attains its least upper bound on M and also attains its greatest lower bound. At a point at which a bound is attained the following equality

$$(y, x, 1) = \nabla(x, y, z) = \lambda \nabla g(x, y, z) = 2\lambda(x, y, z). \tag{2}$$

We have now four equations (1), $1 = 2\lambda z$, $y = 2\lambda x$ and $x = 2\lambda y$. The last equation implies that $y^2 + x^2 = 4\lambda^2(x^2 + y^2)$ so $\lambda = \pm\frac{1}{2}$, we know that $\lambda \neq 0 \neq z$ since $1 = 2\lambda z$. If $2\lambda = 1$ the $z = 1$ so $x^2 + y^2 = 0$ i.e. $x = 0 = y$ and $f(0, 0, 1) = 1$. If $2\lambda = -1$ the $z = -1$ so $x^2 + y^2 = 0$ i.e. $x = 0 = y$ and $f(0, 0, -1) = -1$. This implies that $\sup_M f = 1$ and $\inf_M f = -1$. The bounds are found.

Another method. It is well known that $-\frac{1}{2}(x^2 + y^2) \leqslant xy \leqslant \frac{1}{2}(x^2 + y^2)$ for all real x, y. The left hand-side side inequality follows from the inequality $0 \leqslant \frac{1}{2}(x + y)^2$, the right-hand side from the inequality $0 \leqslant \frac{1}{2}(x - y)^2$. This prove also that the left-hand side inequality becomes equality only for $x = -y$ while the right-hand side inequality turns into equality for $x = y$. Knowing this we write $xy + z \leqslant \frac{1}{2}(x^2 + y^2) + z = \frac{1}{2}(1 - z^2) + z = -\frac{1}{2}(z - 1)^2 + 1 \leqslant 1$, The equality holds only for $z = 1$. In the same way we show that $xy + z \geqslant -1$ and that this equation holds only for $z = -1$. We should keep in mind that $-1 \leqslant z \leqslant 1$. The bounds are found.

One more method. Since $x^2 + y^2 + z^2 = 1$ there exist numbers α and $beta$ such that $x = \cos\alpha\cos\beta$, $y = \cos\alpha\sin\beta$ and $z = \sin\alpha$. Then

$$xy + z = \cos^2\alpha\cos\beta\sin\beta + \sin\alpha = \tfrac{1}{2}\cos^2\alpha\sin(2\beta) + \sin\alpha \leqslant \tfrac{1}{2}\cos^2\alpha + \sin\alpha =$$
$$= \tfrac{1}{2}(1 - \sin^2\alpha) + \sin\alpha = 1 - \tfrac{1}{2}(1 - \sin\alpha)^2 \leqslant 1 \text{ and}$$
$$xy + z = \cos^2\alpha\cos\beta\sin\beta + \sin\alpha = \tfrac{1}{2}\cos^2\alpha\sin(2\beta) + \sin\alpha \geqslant -\tfrac{1}{2}\cos^2\alpha + \sin\alpha =$$
$$= -\tfrac{1}{2}(1 - \sin^2\alpha) + \sin\alpha = -1 + \tfrac{1}{2}(1 + \sin\alpha)^2 \leqslant 1.$$

This is possible since it is easy to parametrize the unit sphere. The Lagrange method allows us to solve the problem without using a specific parametrization which usually is not as easy to find as for the sphere. $\square$

Example 8.2 We go to professor Warhurst example from page 39 of his notes. Let $f(x, y, z) = x + z$ be constrained by the two equations:

$$0 = g_1(x, y, z) = x^2 + y^2 + z^2 - 2 \quad \text{and} \quad 0 = g_2(x, y, z) = x + 2y + 3z - 1. \tag{3}$$

We have $\nabla g_1(x,y,z) = (2x,2y,2z)$ and $\nabla g_2(x,y,z) = (1,2,3)$. These vectors are linearly dependent iff $\begin{vmatrix} x & y \\ 1 & 2 \end{vmatrix} = 0$ and $\begin{vmatrix} x & z \\ 1 & 3 \end{vmatrix} = 0$ and $\begin{vmatrix} y & z \\ 2 & 3 \end{vmatrix} = 0$. They are linearly dependent iff $2x = y$ and $3x = z$ and $3y = 2z$. Due to constraints we have $1 = x + 2x + 3x = 6x$ and $2 = x^2 + 4x^2 + 9x^2 = 14x^2$, a contradiction. We proved that the two gradients are linearly independent for all (x,y,z) for which $g_1(x,y,z) = 0 = g_2(x,y,z)$. Lagrange method can be applied.

The set M defined by the equations (3) is closed and bounded i.e. compact so the $sup_M f$ and $inf_M f$ are values of $f_{|M}$. Therefore there exist numbers λ_1 and λ_2 such that

(4) $\quad 1 = \frac{\partial f}{\partial x} = \lambda_1 \frac{\partial g_1}{\partial x} + \lambda_2 \frac{\partial g_2}{\partial x} = 2x\lambda_1 + \lambda_2$ and

(5) $\quad 0 = \frac{\partial f}{\partial y} = \lambda_1 \frac{\partial g_1}{\partial y} + \lambda_2 \frac{\partial g_2}{\partial y} = 2y\lambda_1 + 2\lambda_2$ and

(6) $\quad 1 = \frac{\partial f}{\partial z} = \lambda_1 \frac{\partial g_1}{\partial z} + \lambda_2 \frac{\partial g_2}{\partial z} = 2z\lambda_1 + 3\lambda_2$.

We may get rid of λ_1 and λ_2. Multiply the first equation by 2 and subtract the second equation from the result: $2 = 2\lambda_1(2x - y)$. Then play the same game with equations (4) and (6): $2 = 2\lambda_1(3x - z)$. Therefore $\lambda_1 \neq 0$ so $2x - y = 3x - z$ i.e.

$$x + y - z = 0. \tag{7}$$

Subtract this equation from $x+2y+3z = 1$. We get $y+4z = 1$ i.e. $y = 1-4z$. Then we obtain $x = z-y = -1+5z$. Therefore $2 = x^2+y^2+z^2 = (-1+5z)^2+(1-4z)^2+z^2 = 2-18z+42z^2$ so either $z = 0$ or $z = \frac{18}{42} = \frac{3}{7}$. In the first case we obtain $(x,y,z) = (-1,1,0)$ and $x+z = -1$. In the second case we get $(x,y,z) = (\frac{8}{7}, -\frac{5}{7}, \frac{3}{7})$ and $x+z = \frac{10}{7}$. We proved that $\sup_M f = \frac{10}{7}$ and $\inf_M f = -1$.

This was just to show everybody that sometimes there are many ways of solving the problem. Sometimes the amount of computational work depends on the way (not in this case). If we look at the problem purely mathematically then λ_1 and λ_2 are just auxiliary unknowns. Therefore the author of this note does not pay too much attention to them. In economy they have some meaning. $\square$

Example 8.3 Let us consider a 3×3 determinant $\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$. We are going to prove that

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \leqslant \sqrt{x_{11}^2+x_{12}^2+x_{13}^2} \cdot \sqrt{x_{21}^2+x_{22}^2+x_{23}^2} \cdot \sqrt{x_{31}^2+x_{32}^2+x_{33}^2}. \tag{8}$$

We start with an observation that if the numbers x_{11}, x_{12}, x_{13} are multiplied by a number $t \geqslant 0$ then both sides of (8) are multiplied by t. The same happens to both sides of the inequality (8) if the second or the third row is multiplied by $t \geqslant 0$. We may multiply the first row by $\frac{1}{\sqrt{x_{11}^2+x_{12}^2+x_{13}^2}}$, the second row by $\frac{1}{\sqrt{x_{21}^2+x_{22}^2+x_{23}^2}}$ and the third row by $\frac{1}{\sqrt{x_{31}^2+x_{32}^2+x_{33}^2}}$. This means that it suffices to prove (8) under the hypothesis

$$0 = x_{11}^2+x_{12}^2+x_{13}^2-1 = x_{21}^2+x_{22}^2+x_{23}^2-1 = x_{31}^2+x_{32}^2+x_{33}^2-1. \tag{9}$$

This means that we are looking for the maximal value of the function

$$f(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \tag{10}$$

under the three constraints (9). We may apply the Lagrange method. To do it we have to find the partial derivatives $\frac{\partial f}{\partial x_{ij}}$ for all nine pairs i,j with $i,j \in \{1,2,3\}$. Let us expand the determinant with respect to the second row:

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = -x_{21} \begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} + x_{22} \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} - x_{23} \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}. \tag{11}$$

This immediately implies that

$$\frac{\partial f}{\partial x_{21}} = - \begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix}, \qquad \frac{\partial f}{\partial x_{22}} = \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix}, \qquad \frac{\partial f}{\partial x_{23}} = - \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}$$

We can do the same thing with two other rows to obtain the result

$$\nabla f(x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}) = \begin{pmatrix} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \\ -\begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \\ \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \end{pmatrix}$$

We wrote ∇f as a matrix because it is easier to understand what is going on. We obtained the matrix consisting of cofactors of the entries of the determinant. If we write in the same way the gradients of functions (9) we obtain

$$\begin{pmatrix} 2x_{11} & 2x_{12} & 2x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 2x_{21} & 2x_{22} & 2x_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2x_{31} & 2x_{32} & 2x_{33} \end{pmatrix}$$

. These matrices or vectors in $\mathbb{R}^9$ are linearly independent because the vectors $(2x_{11}, 2x_{12}, 2x_{13})$, $(2x_{21}, 2x_{22}, 2x_{23})$ and $(2x_{31}, 2x_{32}, 2x_{33})$ have length 2 so if

$$c_1 \begin{pmatrix} 2x_{11} & 2x_{12} & 2x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 0 \\ 2x_{21} & 2x_{22} & 2x_{23} \\ 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2x_{31} & 2x_{32} & 2x_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the $c_1 = c_2 = c_3 = 0$. Lagrange theorem tells us that for each matrix $X := \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

for which the determinant is maximal (or minimal) there are numbers λ_1, λ_2 and λ_3 such that

$$\begin{pmatrix} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \\ -\begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \\ \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \end{pmatrix} =$$

$$= \lambda_1 \begin{pmatrix} 2x_{11} & 2x_{12} & 2x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & 0 \\ 2x_{21} & 2x_{22} & 2x_{23} \\ 0 & 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2x_{31} & 2x_{32} & 2x_{33} \end{pmatrix}$$

If we multiply this equation from the right by $\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}$ then we will obtain

$$\begin{pmatrix} \det(X) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 \\ 2(x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23}) \\ 2(x_{11}x_{31} + x_{12}x_{32} + x_{13}x_{33}) \end{pmatrix}.$$

The result is that $2\lambda_1 = \det(X)$, $x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} = 0$ and $x_{11}x_{31} + x_{12}x_{32} + x_{13}x_{33} = 0$. In the same way (multiplying the equation by two other rows changed into a vertical vector) we prove that $2\lambda_2 = \det(X)$, $2\lambda_3 = \det(X)$ and $x_{21}x_{31} + x_{22}x_{32} + x_{23}x_{33} = 0$. From the obtained equations it follows that $X \cdot X^T = I$. This shows that $1 = \det(I) = \det(X) \cdot \det(X^T) = \det(X)^2$. We proved that if f attains its least upper bound then $\sup(f) = 1$. But obviously our constraints define the bounded set ($|x_{ij}| \leqslant 1$ and closed i.e.compact. Of course there are infinitely many matrices satisfying conditions (9), as many as real numbers. □

Remark 8.4 The dimension 3 is irrelevant. One can prove this the inequality (8) for arbitrary dimension. In fact the above proof works. Many mathematicians call this theorem Hadamard inequality after great French mathematician Jacques Hadamard. It has a geometrical meaning: the volume of the parallelepiped is not greater that the product of lengths of the the edges that have a common vertex. This is just an information. □

Few problems

1. Let $A = \{(x, y) \colon \quad x^2 - y^2 = 1\}$. Find $\sup f$ and $\inf f$, for $f(x, y) = xy$ on A.
Let $B = \{(x, y, z) \colon \quad 7x + 5y + 7z = 1260 \quad \text{and} \quad 7x + y + 7z = 1512\}$. Compute $\sup g$ and $\inf g$ if $g(x, y, z) = 5x^2 + 8xy + 5y^2 + 2z^2$ on the set B.
Solution. Let us start with the set A. We have $x^2y^2 = x^2(x^2 - 1)$ It is clear that to any x with $|x| > 1$ one can assign y so that $x^2 - y^2 = 1$ so $(x, y) \in A$. It is enough to set $y = \pm\sqrt{x^2 - 1}$. The quantity $x^2(x^2 - 1)$ may as big as we can imagine this proves that $\sup\{(xy \colon \quad (x, y) \in A\} = \infty$ and at the same time that $\inf\{(xy \colon \quad (x, y) \in A\} = -\infty$ because $(x, y) \in A$ iff $(-x, y), (x, -y), (-x, -y) \in A$.

Now we shall investigate the second part of this problem. From the two equations that define the set B we get $4y = 1260 - 1512 = -252$ thus $y = -63$. This implies that $7x+7z = 1512+63 = 1575$ so $x+z = 225$. Therefore $g(x,y,z) = 5x^2+8xy+5y^2+2z^2 = $ $=5x^2-504x+5\cdot 3969+5z^2 = 5x^2-504x+19845+2(225-x)^2 = 7x^2-1404x+121,095$. This proves that g is unbounded from above i.e. $\sup\{g(x,y,z)\colon (x,y,z)\in B\} = \infty$. The minimal value is attained for $x = \frac{1404}{14} = 100\frac{2}{7}$. Then $z = 225 - 100\frac{2}{7} = 124\frac{5}{7}$, $y = -63$, so the minimal value of g on the set B is $g(100\frac{2}{7}, -63, 124\frac{5}{7}) = 50694\frac{3}{7}$. We solved the problem without any university knowledge.

Solution 2. Now we show a different solution. Differentiating we get $\nabla(7x+5y+7z-1260) = (7,5,7)$ and $\nabla(7x+y+7z-1512) = (7,1,7)$. The vectors $(7,5,7),(7,1,7)$ are linearly independent therefore the system of the equations defines a manifold (in this case it is a straight line) so can apply Lagrange method. If g attains its maximal or minimal value at a point (x,y,z) them there exist numbers λ_1,λ_2 such that

$$\nabla g(x,y,z) = (10x+8y, 8x+10y, 4z) = \lambda_1(7,5,7)+\lambda_2(7,1,7).$$

We have five equations and five unknowns: $x,y,z,\lambda_1,\lambda_2$. We can start as in the previous solution. So we know that $y=-63$ and $x+z = 225$. This implies that $10x-504 = 7\lambda_1+7\lambda_2$, $8x-630 = 5\lambda_1+\lambda_2$ and $4(225-x) = 7\lambda_1+7\lambda_2$. Therefore $10x-504 = 4(225-x)$ so $14x = 1404$ i.e. $x = \frac{1404}{4} = \frac{702}{7} = 100\frac{2}{7}$. If the function g attains an extreme value it happens for $x = 100\frac{2}{7}$, $y=-63$ and $z = 225-x = 225-100\frac{2}{7} = 124\frac{5}{7}$. One might think that we are almost done but we are not. First of all we have found one point. More important question is: is it a candidate for max or for min and do extreme values exist. The domain is not a compact set so nothing guarantees existence of maximal or minimal value. . It is not hard to notice that $\sup_B g = \infty$. The function g is a quadratic polynomial in the variables x,y,z. The matrix $\begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is positively defined because

$$5>0, \qquad \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 16 > 0 \qquad \text{and} \qquad \begin{vmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 32 > 0.$$

By the Sylvester criterion this implies that the function g attains its minimal value at $(0,0,0)$ and only at this point. We can write $g(tx,ty,tz) = t^2 g(x,y,z)$ for all $t\in\mathbb{R}$. Let $m = \inf\{g(x,y,z)\colon \quad x^2+y^2+z^2 = 1$. This is a value at some point of the unit sphere so $m>0$ (the only point at which $g(x,y,z)=0$ is $(0,0,0)$). Therefore

$$g(x,y,z) = (x^2+y^2+z^2)g\Big(\tfrac{x}{\sqrt{x^2+y^2+z^2}}, \tfrac{y}{\sqrt{x^2+y^2+z^2}}, \tfrac{z}{\sqrt{x^2+y^2+z^2}}\Big) \geqslant m(x^2+y^2+z^2).$$

This implies that if $x^2+y^2+z^2 \geqslant \frac{1\,000\,000}{m}$ then

$$g(x,y,z) \geqslant 1\,000\,000 > 62500 = (5+8+5+2)\cdot 125^2 > g(100\tfrac{2}{7}, -63, 124\tfrac{5}{7}),$$

because $|100\frac{2}{7}|, |-63|, |124\frac{5}{7}| \leqslant 125$.

Let $D = \{(x,y,z)\colon \quad 7x+5y+7z = 1260, 7x+y+7z = 1512, x^2+y^2+z^2 \leqslant \frac{1000000}{m}\}$. D is a subset of B. It is closed and bounded therefore the function g which is continuous attains its minimal value on D at some point (x,y,z) with $x^2+y^2+z^2 < \frac{1\,000\,000}{m}$ because it is at most $g(100\frac{2}{7}, -63, 124\frac{5}{7})$. This proves that $g(100\frac{2}{7}, -63, 124\frac{5}{7}) = \min_D g = \min_B g$. The solution is now complete.

Short summary: the only point of B at which the function could attain its minimal value is found from Lagrange equations. The minimal value exists because outside certain ball centered at the origin the values of the function are huge and in the ball we can apply the Weierstrass maximal/minimal theorem.

2. Let $A = \{(x, y, z): \quad x^2 + y^2 - z = 0 \text{ and } x + y + z = 12\}$. Find the points in the set A at which the function $x^2 + y^2 + z^2$ attains its maximal and minimal values.

Solution. If $(x, y, z) \in A$ then $0 = x^2 + y^2 - (12 - x - y) = (x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 - 12.5$ so $|x + \frac{1}{2}| \leqslant \sqrt{12.5}$ and $|y + \frac{1}{2}| \leqslant \sqrt{12.5}$ so the set A is bounded. It is also closed. Be the Weierstrass theorem there exist points $\mathbf{p}, \mathbf{q} \in A$ such that $f(\mathbf{p}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{q})$ for each point $\mathbf{x} \in A$. We compute $\nabla(x^2 + y^2 - z) = (2x, 2y, -1)$ and $\nabla(x + y + z) = (1, 1, 1)$. These gradients are linearly dependent iff $\begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} 2x & -1 \\ 1 & 1 \end{vmatrix} = 0$ and $\begin{vmatrix} 2y & -1 \\ 1 & 1 \end{vmatrix} = 0$ and therefore $x = y = \frac{1}{2}$ so $z = x^2 + y^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$ but then $x + y + z = \frac{3}{2} \neq 12$. We proved that all points of A the gradients are linearly independent. Therefore at $\mathbf{p}$ nad at $\mathbf{q}$ Lagrange equations must be satisfied. Thus there exist λ_1, λ_2 such that

$$\begin{cases} x^2 + y^2 - z &= 0 \\ x + y + z &= 12 \\ 2x &= \lambda_1 \cdot 2x + \lambda_2 \\ 2y &= \lambda_1 \cdot 2y + \lambda_2 \\ 2z &= -\lambda_1 + \lambda_2 \end{cases} \tag{1}$$

From the third and the fourth equations it follows that $x - y = \lambda_1(x - y)$. From the last two equations it follows $2(y - z) = \lambda_1(2y + 1)$. If $\lambda_1 = 0$ then $x = y = z$ so $x = y = z = 4$ contrary to $x^2 + y^2 - z = 0$. Thus $\lambda_1 \neq 0$. Therefore $(x - y)(2y + 1) = 2(y - z)(x - y)$. Either $x = y$ or $2y + 1 = 2y - 2z$. If $x = y$ then $2x^2 = z$ and $2x + z = 12$ so $2x^2 = 12 - 2x$ so $x^2 + x = 6$ and either $x = 2$ or $x = -3$. In the first case $x = y = 2$ and $z = 8$. In the second case $x = y = -3$ and $z = 18$. If $2y + 1 = 2y - 2z$ then $z = -\frac{1}{2}$ so $x^2 + y^2 = -\frac{1}{2}$, a contradiction. Therefore maximal and minimal values are attained at the points $(2, 2, 8)$ and $(-3, -3, 18)$. The result is $\min(x^2 + y^2 + z^2) = 4 + 4 + 64 = 72$ and $\max(x^2 + y^2 + z^2) = 9 + 9 + 324 = 342$. $\square$

Solution 2. If we treat z as a parameter the system

$$\begin{cases} x^2 + y^2 &= z \\ x + y &= 12 - z \end{cases} \tag{2}$$

has a real solution iff an equation $0 = x^2 + (12 - x - z)^2 - z = 2x^2 - 2x(12 - z) + (12 - z)^2 - z$ has a real root. This happens when

$0 \leqslant (12 - z)^2 - 2(12 - z)^2 + 2z = -(z - 12)^2 + 2(z - 12) + 24 = -(z - 12 - 1)^2 + 25 =$
$= (5 - z + 13)(5 + z - 13) = (18 - z)(z - 8)$ so iff $8 \leqslant z \leqslant 18$.

We have $x^2+y^2+z^2=z+z^2$. Therefore $72=8+8^2\leqslant x^2+y^2+z^2\leqslant 18+18^2=342$ and we are done without derivatives at all. $\square$

3. Let $A=\{(x,y,z)\colon\quad x^2+y^2+z^2=2,\ xy+yz+zx+1=0\}$. Find $\sup f$ and $\inf f$ if $f(x,y,z)=2x+2y-3z$ on the set A.

Solution. The following formulas hold $\nabla(x^2+y^2+z^2-2)=(2x,2y,2z)$ and $\nabla(xy+yz+zx+1)=(y+z,x+z,x+y)$. They are linearly dependent iff the following three equations are satisfied

$$\begin{vmatrix} 2x & 2y \\ y+z & x+z \end{vmatrix}=0,\qquad \begin{vmatrix} 2x & 2z \\ y+z & x+y \end{vmatrix}=0,\qquad \begin{vmatrix} 2y & 2z \\ x+z & x+y \end{vmatrix}=0.$$

This means that $0=x(x+z)-y(y+z)=(x-y)(x+y+z)$, $0=x(x+y)-z(y+z)=$ $=(x-z)(x+y+z)$, $0=y(x+y)-z(x+z)=(y-z)(x+y+z)$. We have $(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=2-2=0$. The gradients are linearly dependent at all points of the set A. Therefore it looks like we are unable to apply Lagrange theorem. But we may notice that the equations $x^2+y^2+z^2-2=0$ and $x+y+z=0$ define the same set A. This changes the situation because $\nabla(x+y+z)=(1,1,1)$. The vectors $(2x,2y,2z)$ and $(1,1,1)$ are linearly dependent iff $x=y=z$ but such points are not in A. Now we can use the Lagrange equations.

$$\begin{aligned} x^2+y^2+z^2-2 &= 0\\ x+y+z &= 0\\ 2 &= 2\lambda_1 x+\lambda_2\\ 2 &= 2\lambda_1 y+\lambda_2\\ -3 &= 2\lambda_1 z+\lambda_2 \end{aligned}\tag{3}$$

We solve this system. Subtract the fourth equation from the third one: $0=2\lambda_1(x-y)$, then the fifth equation from the fourth one: $5=2\lambda_1(y-z)$. This implies that $\lambda_1\neq 0$ and $x=y$. Therefore $2x+z=0$ and $0=2x^2+z^2-2=2x^2+4x^2-2$ so $x=y=\pm\frac{1}{\sqrt{3}}$ and $z=\mp\frac{2}{\sqrt{3}}$. **If** the extreme values are attained they are $2\frac{1}{\sqrt{3}}+2\frac{1}{\sqrt{3}}-3\frac{-2}{\sqrt{3}}=\frac{10}{\sqrt{3}}$ and $2\frac{-1}{\sqrt{3}}+2\frac{-1}{\sqrt{3}}-3\frac{2}{\sqrt{3}}=\frac{-10}{\sqrt{3}}$. They are attained because the set A is bounded since the distance of any point of the set A from the origin is less than or equal to $\sqrt{2}$ and it is closed i.e. it is compact. The function is continuous so it attains $\sup_A f$ and $\inf_A f$. We proved that $\sup_A f=\frac{10}{\sqrt{3}}$ and $\inf_A f=-\frac{10}{\sqrt{3}}$. $\square$

Solution 2. As in the first solution we start with redefining the set A. $A=\{(x,y,z)\colon\quad x^2+y^2+z^2=1,\ x+y+z=0\}$. We can get rid of z. The first (and now only one) condition is $0=x^2+y^2+(-x-y)^2-2=2(x^2+xy+y^2-1)$. The function is $3x+3y-3(-x-y)=5(x+y)$. The problem now is to find $\max(x+y)$ and $\min(x+y)$ constrained to $x^2+xy+y^2=1$. One can easily show that $-\frac{1}{2}(x^2+y^2)\leqslant xy\leqslant\frac{1}{2}(x^2+y^2)$. Therefore $1=x^2+xy+y^2\leqslant\frac{3}{2}(x^2+y^2)$ and $1=x^2+xy+y^2\geqslant\frac{1}{2}(x^2+y^2)$ so $\frac{2}{3}\leqslant x^2+y^2\leqslant 2$. Both bounds are attained: the left for $x=y=\frac{1}{\sqrt{3}}$ while the right for $x=-y=1$. Now $(x+y)^2=x^2+2xy+y^2=1+xy\leqslant 1+\frac{1}{2}(x^2+y^2)$ equality holds for $x=y$ so $x=y=\pm\frac{1}{\sqrt{3}}$. Then $x+y=\frac{2}{\sqrt{3}}$ or $x+y=-\frac{2}{\sqrt{3}}$. Therefore $\sup_A f=\frac{10}{\sqrt{3}}$ and $\inf_A f=-\frac{10}{\sqrt{3}}$. $\square$

4. Let $H = \{(x, y, z) : \quad x^2 + y^2 - z^2 + 4 = 0\}$. Find a point in the set H the closest to the point $(2, 4, 0)$.

Solution. The set H is closed but unbounded, because it contains all points of the form $(x, 0, \sqrt{x^2+4})$. Let $f(x, y, z) = (x-2)^2 + (y-4)^2 + z^2$ i.e. f is a square of the distance from a point (x, y, z) to the point $(2, 4, 0)$. We want to find the minimum of the function f constrained to H or to show that it does not exist. Since $\nabla(x^2 + y^2 - z^2 + 4) = 2(x, y, -z) \neq (0, 0, 0)$ for $(x, y, z) \in H$ and $\nabla f(x, y, z) = 2(x-2, y-4, z)$ there exists a number λ such that $2(x-2, y-4, z) = 2\lambda(x, y, -z)$ for a point (x, y, z) at which the minimum of f is attained. Since $(2, 4, 0) \neq H$ $\lambda \neq 0$. $z \neq 0$ because $(x, y, z) \in H$. Therefore $z = -\lambda z \implies \lambda = -1$. From the equations $x - 2 = \lambda x$ and $y - 4 = \lambda y$ it follows that $x = 1$ and $y = 2$. This proves that if f has a minimal value on H then $\min_H f = f(1, 2, 3) = 14$ or $\min_H f = f(1, 2, -3) = 14$. We can consider the set $\tilde{H} = \{(x, y, z) : \quad x^2 + y^2 - z^2 + 4 = 0, x^2 + y^2 + z^2 \leqslant 10\,000\}$. It is a compact subset of the set H. If $(x, y, z) \in H \setminus \tilde{H}$ then the distance from (x, y, z) to $(2, 4, 0)$ is not less than $100 - \sqrt{2^2 + 4^2 + 0^2} > 75$ so $\inf_H f = \inf_{\tilde{H}} f$. The function f is continuous so it attains its lower bound at some point of the set $\tilde{H}$. The only candidates are $(1, 2 \pm 3)$. Therefore there are two points in H which are the closest to $(2, 4, 0)$. $\square$

Remark 8.5 *The set H is called two sheet (circular) hyperboloid. It arises as a result of rotating a hyperbola defined by the equations $y = 0$ and $z^2 - x^2 = 4$ around the z–axis. It consists of two sheets (connected components) as the hyperbola consists of two branches.* $\square$

5. Let $A = \{(x, y, z) : \quad 5x^2 + 5y^2 - z^2 = 0 \text{ i } x + 2y + 3z = 20\}$. Compute $\sup f$ and $\inf f$ if $f(x, y, z) = x^2 + y^2 + z^2$ on the set A.

*Hint.*You may try to prove that the set A is bounded. The inequalities
$|x + 2y| \leqslant \sqrt{5}\sqrt{x^2 + y^2}, \quad 20 \geq 3|z| - |x + 2y|$ may be helpful.

Solution. Using the hint we obtain $20 \geqslant 3|z| - |x + 2y| \geqslant 3z - \sqrt{5}\sqrt{x^2 + y^2} =$
$= 3\sqrt{5(x^2 + y^2)} - \sqrt{5}\sqrt{x^2 + y^2} = 2\sqrt{5}\sqrt{x^2 + y^2}$. This implies that $20 \geqslant x^2 + y^2$ and therefore $z^2 = 5(x^2 + y^2) \leqslant 100$. We proved that the set A is bounded. It is also closed. Therefore compact. Therefore the (continuous) function $x^2 + y^2 + z^2$ attains its maximal and minimal values on A. The following formulas hold $\nabla(5x^2 + 5y^2 - z^2) = 2(5x, 5y, -z)$ and $\nabla(x + 2y + 3z - 20) = (1, 2, 3)$. These gradients are linearly dependent iff

$$\begin{vmatrix} 5x & 5y \\ 1 & 2 \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} 5x & -z \\ 1 & 3 \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} 5y & -z \\ 2 & 3 \end{vmatrix} = 0,$$

thus $2x = y$, $z = -15x$ and $2z = -15y$. By the definition of A we get $x + 4x - 45x = 20$. This shows that $x = -\frac{1}{2}$, $y = -1$ and $z = \frac{15}{2}$. This is impossible because
$5(-\frac{1}{2})^2 + 5(-1)^2 - (\frac{15}{2})^2 \neq 0$. Therefore at all points of the set A the gradients are linearly independent. By Lagrange theorem there exist numbers λ_1, λ_2 such that
$2(x, y, z) = \nabla(x^2 + y^2 + z^2) = \lambda_1 \nabla(5x^2 + 5y^2 - z^2) + \lambda_2 \nabla(x + 2y + 3z) =$
$$= (10\lambda_1 x + \lambda_2, 10\lambda_1 y + 2\lambda_2, -2\lambda_1 z + 3\lambda_2.$$
This implies that $2(2x - y) = 10\lambda_1(2x - y)$ and $2(3x - z) = 2\lambda_1(15x + z)$. If $\lambda_1 = 0$ then $2x - y = 0$ and $3x - z = 0$. Then $0 = 5x^2 + 5y^2 - z^2 = 5x^2 + 20x^2 - 9x^2$ so $x = 0$ and $y = z = 0$. This contradicts the second equation defining A. Therefore $\lambda_1 \neq 0$. This

implies $4\lambda_1(2x-y)(15x+z) = 20\lambda_1(2x-y)(3x-z)$ i.e. $(2x-y)(15x+z) = 5(2x-y)(3x-z)$. Then either $2x = y$ or $15x + z = 5(3x - z)$. In the first case $0 = 5x^2 + 20x^2 - z^2$ hence $z = 5x$ or $z = -5x$. Then $20 = x + 2y + 3z = 20x$ so $x = 1$, $y = 2$ and $z = 5$ or $20 = x + 2y + 3z = -10x$ so $x = -2$, $y = -4$ and $z = 20$. In the second case $z = 0$ and then $x = y = 0$, a contradiction to $20 = x+2y+3z$. $f(-2,-4,20) = 4+16+400 = 420$, $f(1,2,5) = 1+2+25 = 30$. This proves that $\inf_A f = 30$ and $\sup_A f = 420$. The solution is now complete. $\square$

6. Let $A = \{(x,y)\colon \quad xy = 10\}$. Find $\sup f$ and $\inf f$, if $f(x,y) = 5x + 2y$ on the set A.
Solution. $\nabla(xy - 10) = (y,x) \neq (0,0)$ for $(x,y) \in A$. $\sup(5x + 2y) = \sup(5x + \frac{20}{x}) = \infty$. From the equations $xy = (-x)(-y)$ and $5(-x) + 2(-y) = -5x - 2y$ it follows that $\inf_A(5x + 2y) = -\infty$. We are done.
Let us define $B = \{(x,y)\colon \quad xy = 10 \quad \text{and} \quad x > 0\}$. We shall find $\inf_B f$. If it is attained at some point of B the there is a number λ such that $\nabla(5x + 2y) = \lambda(y,x)$. We have equations $5 = \lambda y$ and $2 = \lambda x$. This implies that $10 = \lambda^2 xy = 10\lambda^2$. This in view of $x > 0$ implies that $\lambda > 0$ so $\lambda = 1$ thus $y = 5$ and $x = 2$ therefore if the smallest value exists it equals to $5 \cdot 2 + 2 \cdot 5 = 20$. We have to prove that the smallest value exist. This can be done by looking at a smaller set than B. If $x > 10$ then $f(xy) > 50$. If $) < x < \frac{1}{25}$ then $f(xy) > 50$. Therefore we can restrict our attention to the set $\hat{B} = \{(x,y) \in B : \frac{1}{25} \leqslant x \leqslant 10\}$. This set is compact so by Weierstrass max/min theorem the continuous function f attains its inf and sup. This proves that $\inf_{\hat{B}} f = \inf_B f = 20$. The end of the problem. $\square$

7. Let $B = \{(x,y,z)\colon \quad x + 6y + 7z = 10 \ \text{ and } \ 4x + 3y + 7z = 12\}$. Find $\sup g$ and $\inf g$ if $g(x,y,z) = x^2 + 9y^2 + 4z^2$ on the set B.
Solution. We start as usually in this kind of problem. $\nabla(x+6y+7z-10) = (1,6,7)$ and $\nabla(4x+3y+7z-12) = (4,3,7)$. These gradients are linearly independent so if $\sup_B g$ or $\inf_B g$ is attained then the Lagrange equations must be satisfied so there exist numbers λ_1, λ_2 such that

$$2(x,9y,4z) = \nabla g = \lambda_1(1,6,7) + \lambda_2(4,3,7) = (\lambda_1 + 4\lambda_2, 6\lambda_1 + 3\lambda_2, 7\lambda_1 + 7\lambda_2).$$

We should solve the system of five equations with unknowns $x, y, z, \lambda_1, \lambda_2$. Adding the first two coordinates we get $2x + 18y = 7\lambda_1 + 7\lambda_2 = 8z$ i.e. $x + 9y - 4z = 0$ or $7x + 63y - 28z = 0$. This implies that $40 = 4(x+6y+7z) + 7x + 63y - 28z = 11x + 87y$ and $48 = 4(4x+3y+7z)+7x+63y-28z = 23x+75y$ so $8 = 12x-12y$ thus $2 = 3x-3y$. This implies that $98 = 40 + 2 \cdot 29 = (11x + 87y) + (87x - 87y) = 98x$. Thus $x = 1$. Therefore $y = \frac{1}{3}$ and $z = \frac{1}{4}(1 + \frac{9}{3}) = 1$. We compute $f\left(1, \frac{1}{3}, 1\right) = 1 + 1 + 4 = 6$.So we know now that **if** the smallest value exists then it is 6. As in few other problems we can look at the values of the function g on the set $\hat{B} = \{(x,yz) \in B\colon \quad x^2 + y^2 + z^2 \leqslant 10$ because outside of $\hat{B}$ the inequality $g(x,y,z) \geqslant 10$ holds. The set $\hat{B}$ is bounded and closed therefore it is compact so the function attains maximal and minimal values on $\hat{B}$. From the above statements it follows right away that $\inf_B g = \inf_{\hat{B}} g = 6$. $\square$

Remark 8.6 The set B above is the intersection of the two planes so it is a straight line. We could have parameterized this line with one variable and find the smallest value of a function of one variable. One way would be to solve the system $x + 6y + 7z = 10$, $4x + 3y + 7z = 12$ of two equations e.g. for x, z. The result would be $x = \frac{2+3y}{3}$, $z = \frac{4-3y}{3}$. The rest of the solution is left to the readers. $\square$

8. Let $A = \{(x, y)\colon \quad x^2 + y^2 = 100\}$. Find $\sup f$ and $\inf f$ if $f(x, y) = 3x + 4y$ on the set A.
Solution. $\nabla(x^2 + y^2 - 100) = 2(x, y) \neq (0, 0)$ for $(x, y) \in A$. The set A is compact so the function f attains maximal and minimal value on this set. Lagrange theorem implies that for each point at which maximal or minimal value is attained there exist a number λ such that $(3, 4) = \lambda\nabla(x^2 + y^2 - 100) = 2\lambda(x, y)$. This implies that $25 = 4\lambda^2(x^2 + y^2) = 400\lambda^2$ so $\lambda = \pm\frac{1}{4}$. Then either $(x, y) = (6, 8)$ or $(x, y) = (-6, -8)$. This proves that $\max_A f = 3 \cdot 6 + 4 \cdot 8 = 50$ and $\min_A f = 3 \cdot (-6) + 4 \cdot (-8) = -50$. We are done. $\square$

9. Find sup and inf of the distances of points $(x, y) \in B = \{(x, y)\colon \quad \frac{x^2}{4} + \frac{y^2}{9} = 1\}$ from the point $(0, \frac{4}{3})$.
Solution. Let $f(x, y) = x^2 + (y - \frac{4}{3})^2$ so $f(x, y)$ is a square of the distance from (x, y) to $(0, \frac{4}{3})$. We shall find the smallest and the largest value of f constrained to B. The extreme values are attained because the set B is compact. $\nabla(\frac{x^2}{4} + \frac{y^2}{9} - 1) = (\frac{x}{2}, \frac{2y}{9}) \neq (0, 0)$ for $(x, y) \in B$. If at the point (x, y) an extreme value is attained the there exits a number λ such that $2(x, y - \frac{4}{3}) = \nabla(x^2 + (y - \frac{4}{3})^2) = \lambda(\frac{x}{2}, \frac{2y}{9})$. $\lambda \neq 0$ because $(x, y) \neq (0, \frac{4}{3}) \notin B$. Therefore $\frac{4xy}{9}\lambda = 2(y - \frac{4}{3})\frac{x}{2} \cdot \lambda$ so $4xy = 3x(3y - 4)$ thus $5xy = 12x$. Therefore either $x = 0$ or $y = \frac{12}{5}$. Extreme values may attained only at the points: $(0, 3)$, $(0, -3)$, $(\frac{6}{5}, \frac{12}{5})$ and $(-\frac{6}{5}, \frac{12}{5})$. The corresponding values of the function are: $\frac{25}{9}$, $\frac{169}{9}$ and $\frac{116}{45}$. This proves that the closest to $(0, \frac{4}{3})$ are the points $(\pm\frac{6}{5}, \frac{12}{5})$. Their distance from $(0, \frac{4}{3})$ is $\frac{2}{3}\sqrt{\frac{29}{5}} \approx 1.61$. The farthest is the point $(0, -3)$ at the distance $\frac{13}{3}$. $\square$

10. Let $A = \{(x_1, x_2, \ldots, x_{10})\colon \quad x_1x_2 + x_1x_3 + \cdots + x_9x_{10} = \sum\limits_{1 \leqslant i < j \leqslant 10} x_ix_j = 45,\ x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_{10} \geqslant 0\}$ and $f(x_1, x_2, \ldots, x_{10}) = x_1 + x_2 + \ldots + x_{10}$. Find the biggest and the smallest value of f on the set A or prove that one of them does not exist or both do not exist.
Solution. Notice that the set A is closed and bounded i.e. it is compact. The function f has therefore maximal and minimal values. We shall find them.
Let $g(x_1, x_2, \ldots, x_{10}) = x_1x_2 + x_1x_3 + \cdots + x_9x_{10}$ – there are $\binom{10}{2} = 45$ summands. $\frac{\partial g}{\partial x_1} = x_2 + x_3 + x_4 + \cdots + x_{10}$, $\frac{\partial g}{\partial x_2} = x_1 + x_3 + x_4 + \cdots + x_{10}$ etc. The gradient of g does not vanish at points of the set A had it been $(0, 0, \ldots, 0)$ the following equality $9(x_1 + x_2 + x_3 + \cdots + x_9 + x_{10}) = 0$ holds i.e. $x_1 + x_2 + x_3 + \cdots + x_9 + x_{10} = 0$, therefore $x_1 = (x_1 + x_2 + x_3 + \cdots + x_9 + x_{10}) - (x_2 + x_3 + \cdots + x_9 + x_{10}) = 0$ etc. but $(0, 0, 0 \ldots 0) \neq A$. From Lagrange theorem it follows that if the function f attains max or min at a point $(x_1, x_2, \ldots, x_{10})$ then either at least one of the numbers $x_1, x_2, \ldots, x_{10}$ is 0 or there exists a number λ such that $\nabla f(x_1, x_2, \ldots, x_{10}) = \lambda\nabla g(x_1, x_2, \ldots, x_{10})$. This means that

$$\begin{cases} 1 = \lambda(x_2 + x_3 + x_4 + \cdots + x_9 + x_{10}) \\ 1 = \lambda(x_1 + x_3 + x_4 + \cdots + x_9 + x_{10}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 = \lambda(x_1 + x_2 + x_3 + x_4 + \cdots + x_9) \end{cases} \tag{4}$$

This implies that

$x_2+x_3+x_4+\dots+x_9+x_{10}=x_1+x_3+x_4+\dots+x_9+x_{10}=\ldots=x_1+x_2+x_3+x_4+\dots+x_9$

From these equations it follows that $x_1=x_2=x_3=\ldots=x_{10}$. This implies that $1=x_1=x_2=x_3=\ldots=x_{10}$. There remains a question of points with one or more coordinates equal to 0. Let $A_1=\{(x_2,x_3,x_4,\ldots,x_{10})\colon\quad x_2x_3+x_2x_4+\dots+x_9x_{10}=45\}$. In the same way we define the sets $A_2,A_3,\ldots,A_{10}$. Now we do the same with f on the sets A_i as we have done with f on the set A. The result is that either the extreme value is attained at a point with equal coordinates or at a point with some coordinate equal to 0. Equal coordinates means that they are $\sqrt{\frac{45}{36}}=\frac{\sqrt5}{2}$. The value of f is $9\cdot\frac{\sqrt5}{2}>10$. Now we have to deal with points with two coordinates being 0. This leads to the value $8\cdot\sqrt{\frac{45}{28}}=12\sqrt{\frac57}>\frac92\sqrt5$. We continue with more zeros. If one looks at points with k zeros, $1\leqslant k\leqslant 8$. Then if all $10-k$ coordinates are equal the value of f is $(10-k)\sqrt{\frac{90}{(10-k)(9-k)}}=\sqrt{\frac{90(10-k)}{9-k}}=\sqrt{90\left(1+\frac{1}{9-k}\right)}$. This expression increases as k does it. So its biggest value is attained for $k=8$. This biggest value is $\sqrt{90\left(1+\frac{1}{9-8}\right)}=6\sqrt5$. We proved that $\inf_A f=10$ and $\sup_A f=6\sqrt5$. □

Remark 8.7 All these problems were exam problems at the beginning of the XXI century at Economy Department of University of Warsaw. There were many others. Some of them are very easy. Usually it was a part of a problem consisting of an easy question and a harder question. □

Remark 8.8 In the online class on Tuesday the problem 5-th of temat24 was discussed. A function of one variable with the two local maxima was needed. The simplest way of finding such a function is to start with its derivative. It should have three roots and appropriate signs on the intervals between the roots. The simplest example is $x(1-x^2)$. This expression is negative iff $x>1$ or $-1<x<0$. Let us integrate this function and multiply the result by 4. We obtain $2x^2-x^4$. This function has local maxima at ±1 and it has local minimum at 0. Define $f(x,y)=2x^2-x^4-y^2$. This function has local maxima at the points $(\pm1,0)$. It has a saddle at $(0,0)$. We have $\nabla f(x,y)=\big(4x(1-x^2),-2y\big)=(0,0)$ iff $(x,y)=(-1,0)$ or $(x,y)=(0,0)$ or $(x,y)=(1,0)$. Therefore there are no other local maxima nor local minima.

We shall give an example of a function which has two local maxima and no other critical point in the plane. Let $g(x,y)=(x,x^2y+e^y)$. The map g is a diffeomorphism of the whole plane onto the set $B=\mathbb{R}^2\setminus\{(0,y)\colon\quad y\leqslant 0\}$ (the plane without one closed vertical half–axis). $\frac{\partial(x^2y+e^y)}{\partial y}=x^2+e^y>0$. This proves that if we fix x the map $y\mapsto x^2y+e^y$ is strictly increasing. It is easy to see that $\lim\limits_{y\to\infty}(x^2y+e^y)=\infty$ and $\lim\limits_{y\to-\infty}(x^2y+e^y)=-\infty$ for $x\neq0$ and $\lim\limits_{y\to-\infty}e^y=0$. This proves that each vertical line except for y–axis is mapped onto itself while y–axis is mapped onto the upper vertical open half–axis. The map is injective (different points are mapped to different points). It is also continuous. We have $Dg(x,y)=\begin{pmatrix}1&0\\2xy&x^2+e^y\end{pmatrix}$ so $\det(Dg(x,y))=x^2+e^y>0$. This proves that the ma p g is locally invertible (inverse function theorem) and the inverse map is C^∞ as g is. This shows that g maps diffeomorphically the plane onto the set B. Now we define a function $F(x,y) = f(g(x,y))$. $g^{-1}(1,0) \approx (1, -0.567143)$ and $g^{-1}(-11,0) \approx (-11, -0.567143)$ are the only critical points of F because $DF(x,y) =$ $=Df(g(x,y))Dg(x,y)$ and $Dg(x,y)$ maps to $(0,0)$ only the vector $(0,0)$ (here we multiply the matrix $Dg(x,y)$ from the left by a vector written horizontally). If one likes an explicit formula for F she/he may see that $F(x,y) = 2x^2 - x^4 - (x^2y + e^y)^2$ and then realize that F has only two critical points. For students who want to be sure they understand correctly these stories.

How to define a C^r function on the whole plane, $r \geqslant 1$, which has three critical points all of them local minima?

Remark 8.9 More of diffeomorphisms from temat XXV and related maps.

$g_1(x,y) = \big(x + y, \frac{2}{\pi}\arctan(\frac{y}{x})\big)$. This maps an interior of the triangle with the vertices $(0,0)$, $(1,0)$ and $(0,1)$ onto the interior of the square with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

$g_2(x,y) = \left(\frac{x}{1-y}, y\right)$. This maps an interior of the triangle with the vertices $(0,0)$, $(1,0)$ and $(0,1)$ onto the interior of the square with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. In both cases it is easy to write formulas for the inverse map. Left to the students.

$g_3(x,y) = (2x - 1, 2y - 1)$ maps the square with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ onto the square with the vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$. Everybody should check it - it is very easy.

$g_4(x,y) = \left(\frac{x}{\sqrt{1-x^2}}, \frac{y}{\sqrt{1-y^2}}\right)$. g_4 maps the square with the vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$ onto the whole plane.

$g_5(x,y) = \big(\tan\frac{\pi x}{2}, \tan\frac{\pi y}{2}\big)$. maps the square with the vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$ onto the whole plane.

$g_6(x,y) = \left(\frac{x}{1-x^2-y^2}, \frac{y}{1-x^2-y^2}\right)$ maps the interior of the unit circle centered at the point $(0,0)$ onto the whole plane. The interior of the unit circle consists of all points (x,y) satisfying $x^2 + y^2 < 1$. On this set the map g_6 is C^∞ (it has derivatives of all orders). We have

$$Dg_6(x,y) = \begin{pmatrix} \frac{1-x^2-y^2+2x^2}{(1-x^2-y^2)^2} & \frac{2xy}{(1-x^2-y^2)^2} \\ \frac{2xy}{(1-x^2-y^2)^2} & \frac{1-x^2-y^2+2y^2}{(1-x^2-y^2)^2} \end{pmatrix} = \frac{1}{(1-x^2-y^2)^4}\begin{pmatrix} 1+x^2-y^2 & 2xy \\ 2xy & 1-x^2+y^2 \end{pmatrix}$$

$\det(DG_6(x,y)) = \frac{(1+x^2-y^2)(1-x^2+y^2)-4x^2y^2}{(1-x^2-y^2)^4} = \frac{1-(x^2-y^2)^2-4x^2y^2}{(1-x^2-y^2)^4} = \frac{1-(x^2+y^2)^2}{(1-x^2-y^2)^4} > 0$ for all (x,y) from the interior of the unit disc. Therefore g_6 is locally invertible (inverse function theorem) and the inverse function g_6^{-1} is C^∞ (as g_6 is). It is globally invertible on the unit disc because it maps each radius onto the infinite ray starting at the origin (the points $(0,0)$, (x,y) and $g_6(x,y)$ lie on one straight line).

$g_7(x,y) = (x^2 - y^2, 2xy)$ is C^∞ and maps the quadrant $\{(x,y)\colon\ \ x > 0, y > 0\}$ onto the half–plane $\{(x,y)\colon\ \ y > 0\}$. Obviously $x > 0, y > 0 \Rightarrow 2xy > 0$. If $u = x^2 - y^2$ and $0 < v = 2xy$ then $u^2 + v^2 = (x^2-y^2)^2 + 4x^2y^2 = (x^2+y^2)^2$. Therefore $x^2 + y^2 = \sqrt{u^2+v^2}$ and $2x^2 = u + \sqrt{u^2+v^2}$ so $x = \sqrt{\frac{1}{2}(u + \sqrt{u^2+v^2})}$. Also $2y^2 = -u + \sqrt{u^2+v^2}$ so $y = \sqrt{\frac{1}{2}(-u + \sqrt{u^2+v^2})}$. We just proved that

$$g_7^{-1}(u,v)=(x,y)=\left(\sqrt{\tfrac{1}{2}(u+\sqrt{u^2+v^2})},\sqrt{\tfrac{1}{2}(-u+\sqrt{u^2+v^2})}\right).$$

This shows that g_7^{-1} is a C^∞ map (all numbers under the square roots are positive).
Let $r>0$ and $g_8(x,y)=\left(\frac{r^2x}{x^2+y^2},\frac{r^2y}{x^2+y^2}\right)$. Clearly if $x^2+y^2=r^2$ then $g_8(x,y)=(x,y)$. g_8 is defined for every point of the plane with one exception: $(0,0)$. It is a C^∞ map. Its inverse is g_8 i.e. $g_8(g_8(x,y))=(x,y)$ for all $x,y)\neq(0,0)$. Let us look at the image of the horizontal line $y=r$. The line consists of points (x,r). $g_8(x,r)=\left(\frac{r^2x}{x^2+r^2},\frac{r^3}{x^2+r^2}\right)$. Now we compute $\left(\frac{r^2x}{x^2+r^2}\right)^2+\left(\frac{r^3}{x^2+r^2}-\frac{r}{2}\right)^2=\frac{4r^4x^2+(r^3-rx^2)^2}{4(r^2+x^2)^2}=\frac{(r^3+rx^2)^2}{4(r^2+x^2)^2}=\frac{r^2}{4}$. This means that the image of the line in question is contained in the circle of radius $\frac{r}{2}$ centered at $(0,\frac{r}{2})$. The only point which is not in this image is $(0,0)$. The reader may conclude that the image of the open half–plane $\{(x,y):\quad y>r$ is the interior of the circle centered at the point $(0,\frac{r}{2})$ of radius $\frac{r}{2}$.
These diffeomorphism allow to find many really different solutions of the problems from temat XXV. $\Box$