Some problems to be solved at home. Choose **4** of them including problem 4, write down your solutions and mail them to me.

Definition 7.1 If U, V are an open subsets of $\mathbb{R}^k$ and $f\colon U \to V$ is a C^r map which maps U onto V and is one–to–one and $f^{-1}\colon V \to U$ is also C^r, $r \geqslant 1$ they we say that f is a C^r–diffeomorphism. $\Box$.

1. Let $Q = \{(x,y) \in \mathbb{R}^2\colon \quad x > 0 \text{ and } y > 0\}$ and $P = \{(x,y) \in \mathbb{R}^2\colon \quad x > 0 \quad \text{or} \quad y > 0\}$. Prove that if $f(x,y) = (x^3 - 3xy^2, 3xy^2 - y^3)$ is a diffeomorphism of the first open quadrant Q onto the set P which consist of points with at least one positive coordinate (it is the compliment of the fourth closed quadrant).

Solution. Unfortunately there is an error in the statement of the problem. I shall discuss the problem as it was stated and later on as I planed to write it.

$$Df(x,y) = \begin{pmatrix} 3x^2 - 3y^2 & -6xy \\ 3y^2 & 6xy - 3y^2 \end{pmatrix}$$

so $\det(Df(x,y)) = (3x^2 - 3y^2)(6xy - 3y^2) + 18xy^3 = 9y(y^3 - x^2y + 2x^3) > 0$ for $(x,y) \in Q$ because if $y \geqslant x$ then $y^3 - x^2y + 2x^3y = y(y+x)(y-x) + 2x^3 \geqslant 2x^3 > 0$ and if $y < x$ then $y^3 - x^2y + 2x^3y = y^3 + x^2(x-y) + x^3 > 0$. This creates a hope for validity of the theorem we have to prove. From the definition of the diffeomorphism it follows that $1 = \det(D(f \circ f^{-1})(f(x,y))) = \det(Df(x,y)) \cdot \det(Df^{-1}(f(x,y)))$ therefore $\det(Df(x,y)) \neq 0$. Let us notice that that $f(tx,ty) = t^3 f(x,y)$ for each $t \in \mathbb{R}$ and each $(x,y) \in Q$. If $(x,y) \in Q$ then there exist numbers $r > 0$ and $\varphi \in (0, \frac{\pi}{2})$ such that $x = r\cos\varphi$ and $y = r\sin\varphi$. First assume that $r = 1$. We have

$$f(\cos\varphi, \sin\varphi) = (\cos^3\varphi - 3\cos\varphi\sin^2\varphi,\ 3\cos\varphi\sin^2\varphi - \sin^3\varphi).$$

This is never $(0,0)$ because if $\cos^3\varphi - 3\cos\varphi\sin^2\varphi = 0$ then either $\cos\varphi = 0$ or $\cos^2\varphi = 3\sin^2\varphi$. In the first case $\sin\varphi = \pm 1$ so $3\cos\varphi\sin^2\varphi - \sin^3\varphi = \mp 1 \neq 0$. In the second case $3\cos\varphi\sin^2\varphi - \sin^3\varphi = \cos^3\varphi - \sin^3\varphi$. If $0 = \cos^3\varphi - \sin^3\varphi$ then $\cos\varphi = \sin\varphi$ so $\cos^2\varphi = \sin^2\varphi = \frac{1}{2}$. This contradicts our hypothesis i.e. the equality $\cos^2\varphi = 3\sin^2\varphi$. Let

$$\varrho(\varphi) = \sqrt{(\cos^3\varphi - 3\cos\varphi\sin^2\varphi)^2 + (3\cos\varphi\sin^2\varphi - \sin^3\varphi)^2}.$$

This is a C^∞ function of φ because the quantity under the square root is always positive. Let $\alpha(\varphi)$ be such a number that the following three conditions hold

$$\cos\alpha(\varphi) = \frac{\cos^3\varphi - 3\cos\varphi\sin^2\varphi}{\varrho(\varphi)}, \quad \sin\alpha(\varphi) = \frac{3\cos\varphi\sin^2\varphi - \sin^3\varphi}{\varrho(\varphi)}, \quad 0 < \alpha(\varphi) < 2\pi.$$

The number $\alpha(\varphi)$ exists because $0 < \varphi < \frac{\pi}{2}$ so $\sin\varphi \neq 0$ and if $\sin\alpha(\varphi) = 0$ then $\sin\varphi = 3\cos\varphi$ but then $\varrho(\varphi)\cos\alpha(\varphi) = \cos^3\varphi - 3\cos\varphi\sin^2\varphi = -26\cos^3\varphi < 0$. This means that the range of f considered on the domain Q contains no points of the form $(x,0)$ with $x \geqslant 0$ and due to this we can choose α as required above. It is not hard to show that α is a C^1 function of φ, in fact it is C^∞. It is enough to use the properties of arcsin and arccos. Another way of proving it is to use the inverse function theorem. Thus we can differentiate α and functions which depend on φ. We have $\varrho\cos\alpha = \cos^3\varphi - 3\cos\varphi\sin^2\varphi$ and $\varrho\sin\alpha = 3\cos\varphi\sin^2\varphi - \sin^3\varphi$. This implies that

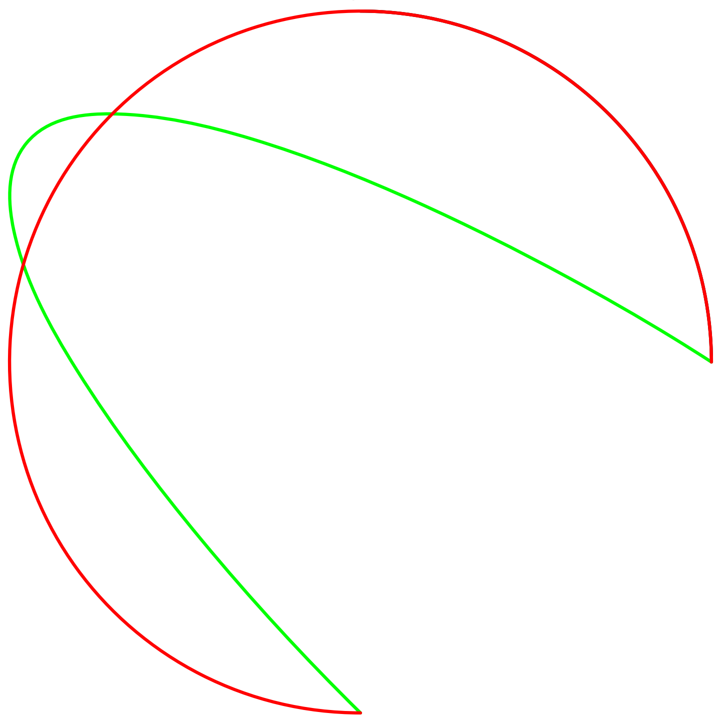
$$\cos\alpha\frac{d\varrho}{d\varphi} - \varrho\sin\alpha\frac{d\alpha}{d\varphi} = -3\cos^2\varphi\sin\varphi + 3\sin^3\varphi - 6\cos^2\varphi\sin\varphi = 3\sin^3\varphi - 9\cos^2\varphi\sin\varphi$$

and

$$\sin\alpha\frac{d\varrho}{d\varphi} + \varrho\cos\alpha\frac{d\alpha}{d\varphi} = -3\sin^3\varphi + 6\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi.$$

We multiply the last equation by $\varrho\cos\varphi$ and the next to the last by $-\varrho\sin\varphi$ and add the results to obtain

$\varrho^2\frac{d\alpha}{d\varphi} = \varrho\cos\alpha(-3\sin^3\varphi + 6\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi) - \varrho\sin\alpha(3\sin^3\varphi - 9\cos^2\varphi\sin\varphi) =$
$= (\cos^3\varphi - 3\cos\varphi\sin^2\varphi)(-3\sin^3\varphi + 6\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi) +$
$+ (3\cos\varphi\sin^2\varphi - \sin^3\varphi)(-3\sin^3\varphi + 9\cos^2\varphi\sin\varphi) =$
$= 3\sin\varphi(2\cos^5\varphi - \cos^4\sin\varphi + 2\cos^3\varphi\sin^2\varphi + \sin^5\varphi) =$
$= 3\sin\varphi(2\cos^3\varphi + \sin\varphi(\sin^4\varphi - \cos^4\varphi)) = 3\sin\varphi(2\cos^3\varphi + \sin\varphi(\sin^2\varphi - \cos^2\varphi)) =$
$= 3\sin\varphi(2\cos^3\varphi + \sin^3\varphi - \sin\varphi\cos^2\varphi) > 0$ because $0 < \cos\varphi < 1$ and $0 < \sin\varphi < 1$. As e result we obtain $\frac{d\alpha}{d\varphi} > 0$. Therefore α is a strictly increasing function of φ. This shows that each infinite ray that starts at $(0,0)$ meets the set $\{(\varrho\cos\alpha, \varrho\sin\alpha)\colon \quad 0 < \varphi < \frac{\pi}{2}\}$ at most at one point. The map f is defined and continuous not only on the set Q but on the whole plane. $f(1,0) = (1,0)$, $f(0,1) = (0,-1)$. This implies that $\lim\limits_{\varphi\searrow 0}\alpha(\varphi) = 0$ and $\lim\limits_{\varphi\nearrow\pi/2}\alpha(\varphi) = \frac{\pi}{2}$. Since α is a continuous function of φ all numbers from the interval $(0\frac{\pi}{2})$ are its values. Therefore on each ray starting at the origin contained in P there is a point from $f(Q)$. This together with the equation $f(rx,ry) = r^3f(x,y)$ proves that $f(Q) = P$. This also proves that f is one–to–one on each ray and maps a ran onto a ray. Therefore f is one–to–one map on Q and its image is P. We did it.



Black is a quarter of the unit circle, green its image under the first map, red its image under the second.

Now we shall solve a problem as it was planned.

This time $f(x,y)=(x^3-3xy^2,3x^2y-y^3)$. Obviously $f(rx,ry)=r^3f(x,y)$ (as above). Now

$$(x^3-3xy^2)^2+(3x^2y-y^3)^2=x^6-6x^4y^2+9x^2y^4+9x^4y^2-6x^2y^4+y^6=$$
$$=x^6+3x^4y^2+3x^2y^4+y^6=(x^2+y^2)^3.$$

We have $f(\cos\varphi,\sin\varphi)=(\cos^3\varphi-3\cos\varphi\sin^2\varphi,3\cos^2\varphi\sin\varphi-\sin^3\varphi)$ and from the above equation it follows that the point $(\cos^3\varphi-3\cos\varphi\sin^2\varphi,3\cos^2\varphi\sin\varphi-\sin^3\varphi)$ lies on the unit circle so there exist α such that

$$(\cos^3\varphi-3\cos\varphi\sin^2\varphi,3\cos^2\varphi\sin\varphi-\sin^3\varphi)=(\cos\alpha,\sin\alpha).$$

Some people (very good in trigonometry) know that $\cos(3\varphi)=\cos^3\varphi-3\cos\varphi\sin^2\varphi$ and $\sin(3\varphi)=3\cos^2\varphi\sin\varphi-\sin^3\varphi$. If someone does know this and she/he does not want to look for the formula in the internet or in books she/he may notice that α is a differentiable function of φ provided that $\alpha\in(0,2\pi)$. Such choice is possible because if $0=3\cos^2\varphi\sin\varphi-\sin^3\varphi$ (this means that $f(\cos\varphi,\sin\varphi)$ lies on x–axis) then either $\sin\varphi=0$ or $3\cos^2\varphi-\sin^2\varphi=0$. But $0<\varphi<\frac{\pi}{2}$ so $0<\sin\varphi<1$ and $0<\cos\varphi<1$. The first possibility has been excluded. In the second case we have $\cos^3\varphi-3\cos\varphi\sin^2\varphi=\cos^3\varphi-9\cos^3\varphi=-8\cos^3\varphi<0$. We proved that $f(\cos\varphi,\sin\varphi)$ lies on x–axis then it lies to the left of the origin, so in such case we define $\alpha(\varphi)=\pi$. The differentiability follows from the differentiability of arccos and arcsin on the interval $(-1,1)$.

The map f is diffeomorphism because it is one–to–one map of Q onto P. $\square$

Remark 7.2 $Df(x,y)=\begin{pmatrix}3x^2-3y^2 & -6xy\\ 6xy & 3x^2-3y^2\end{pmatrix}$ so $\det(Df(x,y))=9(x^2-y^2)^2+36x^2y^2=9(x^2+y^2)^2>0$ for $(x,y)\in Q$. this guarantees that for each point $\mathbf{p}=(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ there a number $r_{\mathbf{p}}$ such that f is o diffeomorphism of the disc of radius $r_{\mathbf{p}}$ centered at $\mathbf{p}$ onto some open subset of $\mathbb{R}^2$. Unfortunately this does not prove the the map is one–to–one on the whole set $\mathbb{R}^2\setminus\{(0,0)\}$. In fact the map is NOT one–to–one e.g. $f(\sqrt{3},1)=(0,8)=f(-\sqrt{3},1)$. The set $\mathbb{R}^2\setminus\{(0,0)\}$ is connected, the derivative $Df(x,y)$ is everywhere invertible ($Df(x,y)$ is an isomorphism for each (x,y)) but the map f is not invertible. This is one of many important differences between one dimension and more of them. $\square$

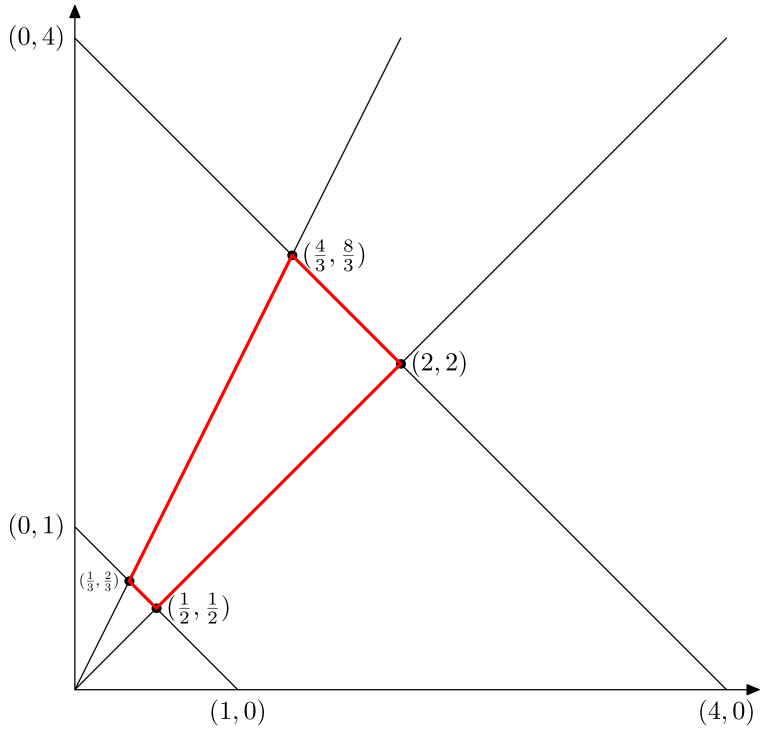
Definition 7.3 $A\times B$ is a set consisting of all ordered pairs (a,b) with $a\in A$ and $b\in B$. For example $\{1,2,3,4\}\times\{4,5\}=\{(1,4),(2,4),(3,4),(4,4),(1,5),(2,5),(3,5),(4,5)\}$, $(0,\infty)\times(0,\infty)$ is the first open quadrant which consists of all pairs of positive real numbers. $\square$.

2. Does there exist a diffeomorphism of the set $\{(x,y):\quad x<y<2x \text{ and } 1<x+y<4\}$ onto

(1) an open square,

(2) the whole plane $\mathbb{R}^2$.

Solution. In both cases the answer is yes. Notice that from the inequality $x < 2x$ it follows that $x > 0$, so the set is contained in the first quadrant ant it is open and bounded. In fact it is a quadrilateral with two paralel sides so it is a trapezium (British English not American English). The vertices of this quadrilateral are $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{2}, \frac{1}{2})$, $(2, 2)$ and $(\frac{4}{3}, \frac{8}{3})$ as you can check. Do it!



Let us define $f(x,y) = \frac{y}{x}, x+y$ and $T = \{(x,y)\colon \quad x < y < 2x \text{ and } 1 < x+y < 4\}$. If $(\frac{y}{x}, x+y) = (u,v)$ then $x = \frac{v}{1+u}$ and $y = \frac{uv}{1+u}$ in other words $f^{-1}(u,v) = \left(\frac{v}{1+u}, \frac{uv}{1+u}\right)$. All numbers in these formulas are positive. To be more precise $1 < u < 2$ and $1 < v < 4$. The image of T is a rectangle, $f(T) = (1,2) \times (1,4)$, and we can make easily a square out of it. Just define $\hat{f}(x,y) = (\frac{3y}{x}, x+y)$. One can see that $\hat{f}(T) = (3,6) \times (1,4)$ so it is a square with side length 3. It is worth to notice that $\hat{f}\big((0,\infty) \times (0,\infty)\big) = \big((0,\infty) \times (0,\infty)\big)$ so the first quadrant is mapped onto itself.

Let us define now $g(x,y) = \big(\tan(\pi(\frac{y}{x} - \frac{3}{2})), \tan(\frac{\pi}{3}(x+y-\frac{5}{2}))\big)$. I claim that $g(T) = \mathbb{R}^2$. The map $u \mapsto \pi(u - \frac{3}{2})$ maps the interval $(1,2)$ onto the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ then tangent maps the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$. The map $v \mapsto \frac{\pi}{3}(v - \frac{5}{2})$ onto the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ which is mapped by tangent onto $(-\infty, \infty)$. We are done. $\square$

3. Prove that there is no diffeomorphism $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that the x–axis is mapped onto the set $\{(x,0)\colon \quad x \geqslant 0\} \cup \{(0,y)\colon \quad y \geqslant 0\}$ i.e. onto the boundary of the first quadrant. *Hint.* How $Df(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^2$ maps tangent vectors?

In this problem we are asked of the existence of a diffeomorphism which maps the straight line onto the union of two rays with initial point $(0,0)$. We shall show that this is not possible, in general the image of a smooth curve under a diffeomorphism is smooth. The union of the two rays is not smooth, there is a corner at $(0,0)$ - these statements are not precise because we never said what is a smooth curve, I wrote hoping that it would give some intuition. Let us assume that there is such a diffeomorphism and let $(\gamma_1(t), \gamma_2(t)) = \gamma(t) = f(t,0)$. There is a point t_0 such that $\gamma(t_0) = (0,0)$. This means that $\gamma_1(t_0) = 0$ and $\gamma_2(t) = 0$. There are two possibilities. Either $\gamma_1(t) = 0$ and $\gamma_2(t) > 0$ for $t > t_0$ – this means that of the first quadrant and therefore the half line $(-\infty, t_0]$ is mapped onto the horizontal part of the boundary that is for $t < t_0$ the equality $\gamma_2(t) = 0$ and the inequality $\gamma_1(t) > 0$ hold. Of course it may happen that the half line $[t_0, \infty)$ is mapped onto the horizontal part of the boundary while the half line $(-\infty, t_0]$ is mapped onto the vertical part of the boundary. In both cases the functions γ_1, γ_2 assume their extreme values at t_0. Therefore $\gamma_1'(t_0) = 0 = \gamma_2'(t_0)$ so $\gamma'(t_0) = (0, 0)$. This is not possible since $\gamma_1(t_0) = \frac{\partial f}{\partial x}(t_0, 0)$ and $\det(Df(t_0, 0)) \neq 0$ so it is not possible a column of the matrix $Df(t_0, 0)$ vanishes. □

4. We consider the system of two equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \end{cases}$$

with unknowns x_1, x_2, x_3, x_4. Under what condition on the coefficients $a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}, b_1, b_2$ the system can solved for x_1, x_2 treating the unknowns x_3, x_4 as parameters.

Solution. The problem is not very well stated. It should be *uniquely solved* instead of *solved.* We shall show the solution with this slightly extended statement to avoid long considerations which are not hard. This is in fact to say that we want a condition that guarantees the system $a_{11}x_1 + a_{12}x_2 = c_1$ and $a_{21}x_1 + a_{22}x_2 = c_2$ is uniquely solvable for x_1, x_2. This is or at least should be very well known to everybody but I show it.

If both equations are satisfied then

$$(a_{11}a_{22} - a_{21}a_{12})x_1 = c_1a_{22} - c_2a_{12} \quad \text{and} \quad (a_{22}a_{11} - a_{21}a_{12})x_2 = c_2a_{11} - c_1a_{21}$$

– the equations have been multiplied by the appropriate numbers and added later on.

If $a_{22}a_{11} - a_{21}a_{12} = 0$ then necessarily $c_1a_{22} - c_2a_{12} = 0 = c_2a_{11} - c_1a_{21}$.

If $a_{11} = a_{12} = 0$ then there is no unknown in the first equation. If in addition $c_1 = 0$ the we have only second equation (the first $0x_1 + 0x_2 = 0$ is satisfied for all $x_1, x_2 \in \mathbb{R}$). If $c_1 \neq 0$ the system has no solution. If $a_{21} \neq 0$ or $a_{22} \neq 0$ the second equation has infinitely many solutions so the has the system in this (stupid) case. The same happens if $a_{21} = 0 = a_{22}$.

Now we assume that $a_{22}a_{11} - a_{21}a_{12} = 0$ and $(a_{11}, a_{12}) \neq (0, 0) \neq (a_{21}, a_{22})$. If $a_{11} \neq 0$ then $a_{22} = \frac{a_{21}}{a_{11}}a_{12}$, so $(a_{21}, a_{22}) = \frac{a_{21}}{a_{11}}(a_{11}, a_{12})$. The system has a solution iff also $c_2 = \frac{a_{21}}{a_{11}}c_2$ but this means that the second equation follows from the first one so the system has infinitely many solutions (for each number x_2 we can find a number x_1 such that the (x_1, x_2) satisfies the first equationso it satisfies the second one.

We proved that if $a_{22}a_{11} - a_{21}a_{12} = 0$ then the system either has no solution or it has infinitely many ot them.

Now assume that $a_{22}a_{11} - a_{21}a_{12} \neq 0$. The it is easy to see that the the pair (x_1, x_2) with $x_1 = \frac{c_1a_{22} - c_2a_{12}}{a_{22}a_{11} - a_{21}a_{12}}$, $x_2 = \frac{c_2a_{11} - c_1a_{21}}{a_{22}a_{11} - a_{21}a_{12}}$ solves the system, just substitute the obtained quantities for x_1 and x_2 in the system. It is clear that this the unique choice of the solution. So the seeked condition is $a_{22}a_{11} - a_{21}a_{12} \neq 0$. □

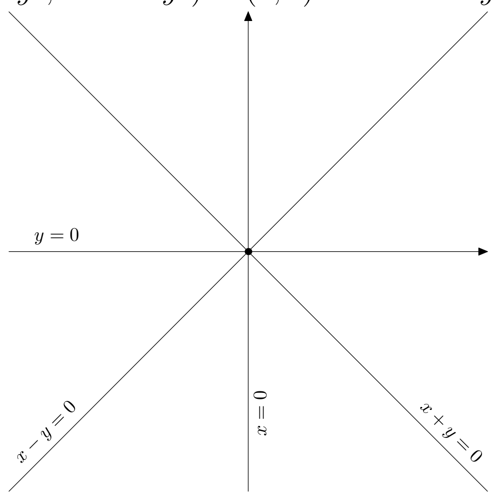
Remark 7.4 Notice that in the implicit function theorem with the equation

$$F(x_1,x_2,x_3,x_4) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 - b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 - b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the condition concides with saying that the matrix $\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}$ is nonsingular i.e. its determinant does not vanish. One may say that the implicit function theorem generalizes the linear algebra theorem but due to hypothesis about derivatives at one point only the theorem becomes local – it says something about existence and uniqueness of the solutions at sufficiently small neighbourhood only. $\square$

5. Find all $c \in \mathbb{R}$ for which the set $\{(x,y): \quad xy = c\}$ an embedded submanifold of $\mathbb{R}^2$.
Solution. Let $f(x,y) = xy - c$. One can see that $\nabla f(x,y) = (y,x)$ so it does not vanish unless $x = 0 = y$. The point $(0,0)$ is in the set iff $c = 0$ so if $c \neq 0$ the set is a manifold b y the theorem that apperas in prof. Warhurst's notes. If $c = 0$ then the equation $xy = c$ is satisfied by all points that lie on the union of x–axis ($y = 0$) and y–axis ($x = 0$). This set is not a manifold because of $(0,0)$. If one looks at any connected neighbourhood of $(0,0)$ then it is a „open cross". If one throws out the point $(0,0)$ it is divided into four connected parts (straight line segments). This cannot happen to one dimensional manifold because locally it is equivalent to an open interval so if a point is thrown away it falls apart into two pieces (connected components). $\square$

6. Draw the set M defined by the equation $xy(x^2 - y^2) = 0$. What points should be removed from M so that the remaining part of M will be an embedded submanifold of $\mathbb{R}^2$. The number of the removed points should be as small as possible.
Solution. Let $f(xy) = xy(x^2 - y^2)$. We have $\nabla f(x,y) = (3x^2y - y^3, x^3 - 3xy^2)$. It is easy to see that $(3x^2y - y^3, x^3 - 3xy^2) = (0,0)$ iff $x = 0 = y$.



This shows that if we remove the origin from the set then it will become a 1–dimensional submanifold of $\mathbb{R}^2$. With $(0,0)$ the set is not a submanifold. It consists of four straight lines that meet each other at $(0,0)$. If we consider any connected neighbourhood of $(0,0)$ in the set and remove the origin from it then it becomes a union of eight disjoint pieces. This is not a property of any open interval: a point divides it into two two disjoint parts (components). This proves that there is only one point to be removed from the set, namely $(0,0)$. $\square$