Some problems to be solved at home. Choose **4** of them, write down your solutions and mail them to me.

**1**. Given a triangle $ABC$. Let $d(P) = PA + PB + PC$ so it is the sum of the distances from $P$ to the vertices of the triangle. Prove that here exists a point $Q$ contained in the triangle $ABC$ such that $d(Q) \leqslant d(p)$ for every point $P$ of the given triangle.

*Solution.* Let us recall that if $Q, R, S$ are points (in the plane) then

$$\|Q - S\|_2 \leqslant \|Q - R\|_2 + \|R - S\|_2.$$

This inequality turns into equality iff $R$ lies on straight line segment with ends $Q, S$. This implies that $\|Q - S\|_2 - \|Q - R\|_2 \leqslant \|R - S\|_2$. Also $\|Q - R\|_2 - \|Q - S\|_2 \leqslant \|S - R\|_2 = \|R - S\|_2$. Therefore $\big|\|Q - S\|_2 - \|Q - R\|_2\big| \leqslant \|R - S\|_2$. Noe let $Q, R$ be any points on the plane. Be the previous inequalities we obtain

$$|d(Q) - d(R)| = \big|\|Q - A\|_2 + \|Q - B\|_2 + \|Q - C\|_2 - \|R - A\|_2 - \|R - B\|_2 - \|R - C\|_2\big| \leqslant$$
$$\leqslant \big|\|Q - A\|_2 - \|R - A\|_2\big| + \big|\|Q - B\|_2 - \|R - B\|_2\big| + \big|\|Q - C\|_2 - \|R - C\|_2\big| \leqslant 3\|Q - R\|_2.$$

This proves that $d$ is a continuous function on the plane. It one thinks of a triangle $ABC$ only then the function is considered on a compact set set. By Weierstrass Maximum Principle it attains minimum at some point of the triangle. QED □

*Remark. The solution contains no indication of how to find the point at which the minimum is attained nor wether there is one such point or more. What is your guess?*

***2***. Find all critical points of the function $f$ and its least upper bound and its greatest lower bound if $f(x, y) = x^2 + y^2(1 + x)^3$ for $(x, y) \in \mathbb{R}^2$.

*Solution.* $\frac{\partial f}{\partial x} = 2x + 3y^2(1+x)^2$, $\frac{\partial f}{\partial y} = 2y(1+x)^3$. At a critical point both partial derivatives should equal to 0. $\frac{\partial f}{\partial y} = 2y(1+x)^3 = 0$ iff $y = 0$ or $x = -1$. It $y = 0$ and $\frac{\partial f}{\partial x} = 2x+3y^2(1+x)^2 = 0$ the $y = 0$. If $x = -1$ and $\frac{\partial f}{\partial x} = 2x + 3y^2(1 + x)^2 = 0$ then $x = 0$ a contradiction. Thus there is only one critical point namely $(0, 0)$. If $x > -1$ then $f(x, y) = x^2 + y^2(1 + x)^3 \geqslant 0$ and $f(x, y) = 0$ iff $x = y = 0$. This means that the function $f$ has local minimum at $(0, 0)$. $f(x, 1) = x^2 + (1 + x)^3 \xrightarrow[x\to\infty]{} +\infty$ and $f(x, 1) = x^2 + (1 + x)^3 \xrightarrow[x\to-\infty]{} -\infty$. This proves that $f$ is unbounded from above so $\sup f = +\infty$ and it is unbounded from below so $\inf f = -\infty$. □

*Remark. One may evaluate second partial derivatives at* $(0, 0)$. $\frac{\partial^2 f}{\partial x^2} = 2 + 6y^2(1 + x)$, $\frac{\partial^2 f}{\partial x \partial y} = 6y(1+x)^2$ *and* $\frac{\partial^2 f}{\partial y^2} = 2(1+x)^3$. *This implies that* $d^2 f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *This matrix is positively defined because its eigenvalues are* $2, 2$ *so they are positive. One may also use the Sylvester theorem or the definition of positively defined matrix that. The last method leads to looking at the expression* $2 \cdot x^2 + 2 \cdot 0 \cdot xy + 2 \cdot y^2 = 2(x^2 + y^2)$. *This positive for all* $(x, y)$ *except for* $x = 0 = y$. *So the matrix is positively defined and the function has a local minimum at* $(0, 0)$. □

***3***. (a) Let $f(x, y) = 6x^5 + 15x^4 - 50x^3 - 90x^2 + \frac{1}{4}(-e^{2y} + x^2(x + 3)^2)^2$ for all $(x, y) \in \mathbb{R}^2$. Find all critical points of $f$, determine their charakter (local minimum, local maximum or saddle), find $\sup f$ and $\inf f$.

(b) Let $g(x, y) = 6x^5 + 15x^4 - 50x^3 - 90x^2 + \frac{1}{4}(-e^{2y} + (x+1)^2(x-2)^2)^2$ for all $(x, y) \in \mathbb{R}^2$. Find all critical points of $g$, determine their charakter (local minimum, local maximum or saddle), find $\sup g$ and $\inf g$.

(c) Let $h(x, y) = 6x^5 + 15x^4 - 50x^3 - 90x^2 + \frac{1}{4}(-e^{2y} + (x+1)^2(x+3)^2)^2$ for all $(x, y) \in \mathbb{R}^2$. Find all critical points of $h$, determine their charakter (local minimum, local maximum or saddle), find $\sup h$ and $\inf h$.

*Solution.* Let $p(x) = 6x^5 + 15x^4 - 50x^3 - 90x^2$. We have

$p'(x) = 30x^4 + 60x^3 - 150x^2 - 180x = 30x(x^3 + 2x^2 - 5x - 6) = 30x(x + 1)(x - 2)(x + 3)$.

This implies that $p'(x) > 0$ iff $x \in (2,\infty) \cup (-1,0) \cup (-\infty,-3)$. This implies that the function $p$ increases (strictly) on each of the intervals $(-\infty,-3]$, $[-1,0]$, $[2,\infty)$ and decreases (strictly) on each of the intervals $[-3,-1]$ and $[0,2]$.

Let $\alpha(x,y) = p(x)$. The function $\alpha$ has local minima at points $(-1,y)$, $(2,y)$ independently of $y$ and local maxima at points $(-3,y)$, $(0,y)$. All are of course improper.

Let $\varphi_1(x,y) = \frac{1}{4}(-e^{2y} + x^2(x+3)^2)^2$. We have $\frac{\partial\varphi_1}{\partial y} = -e^{2y}(-e^{2y} + x^2(x+3)^2) = 0$ iff $e^{2y} = x^2(x+3)^2$. Now $\frac{\partial\varphi_1}{\partial x} = p'(x) + x(x+3)(2x+3)(-e^{2y} + x^2(x+3)^2)$. Therefore

$$\nabla f(x,y) = \big(p'(x) + \tfrac{\partial\varphi_1}{\partial x}, \tfrac{\partial\varphi_1}{\partial y}\big) = (0,0)$$

iff $e^{2y} = x^2(x+3)^2$ and $p'(x) = 0$ i.e. either $x=-3$ or $x=-1$ or $x=0$ or $x=2$. Obviously $\varphi_1(x,y) = \frac{1}{4}(-e^{2y} + x^2(x+3)^2)^2 \geqslant 0$ everywhere so at all points $(x,y)$ at which $-e^{2y} + x^2(x+3)^2 = 0$ it has an absolute minimum. Since $0 < e^{2y}$ it follows that either $x=-1$ or $x=2$. We proved that $f$ has 2 critical points $(-1,\ln 2)$ and $(2,\ln 10)$. Since the functions $\varphi_1$ and $\alpha$ have local minima at $(2,\ln 10)$ their sum $f$ has a local minimum at $(2,\ln 10)$, too. Since this is an isolated critical point of $f$ the local minimum is proper. This means that if $x \approx 2$ and $y \approx \ln 10$ and $(x,y) \neq (2,\ln 10)$ then $f(x,y) > f(2,\ln 10)$. We do not explain how close to $(2,\ln 10)$ the point $(x,y)$ should be because this information is not necessary for the solution of the problem.

All we wrote above applies to the point $(-1,\ln 2)$ The main point is that the polynomial $p$ has local minima at $-1$ and at $2$.

Let $\varphi_2(x,y) = \frac{1}{4}(-e^{2y} + (x+1)^2(x-2)^2)^2$, so $g(x,y) = p(x) + \varphi_2(x,y) = \alpha(x,y) + \varphi_2(x,y)$. The situation is a little bit different from the one discussed above. The difference is that now the critical points are $(0,\ln 2)$ and $(-3,\ln 15)$. At the points 0 and 3 the polynomial $p$ has local maxima so does the function $\alpha$ at the points $(0,\ln 2)$ and $(-3,\ln 15)$. But the function $\varphi_2$ has local minima at the points $(0,\ln 2)$ and $(-3,\ln 15)$. We shall prove that the function $g$ has saddles at the points $(0,\ln 2)$ and $(-3,\ln 15)$. If one moves away from $(0,\ln 2)$ along the line $x=0$ ($y$–axis) then $\varphi_2(x,y)$ becomes strictly positive so $f(x,y) > f(0,\ln 2)$. If one moves away from the point $(0,\ln 2)$ along the graph of the function $y = \frac{1}{2}\ln((x+1)^2(x-2)^2) = \ln((x+1)(2-x))$ w otoczeniu punktu 0 then the function $f$ decreases because $\varphi_2$ does it and the value of $\alpha$ is 0 at all points of the graph. The same argument applies to the point $(-3,\ln 15)$.

Let $\varphi_3(x,y) = \frac{1}{4}(-e^{2y} + (x+1)^2(x+3)^2)^2$, so

$$h(x,y) = \alpha(x,y) + \varphi_3(x,y) = 6x^5 + 15x^4 - 50x^3 - 90x^2 + \tfrac{1}{4}(-e^{2y} + (x+1)^2(x+3)^2)^2.$$

Applying the same arguments as above we discover the critical points $(0,\ln 3)$ and $(2,\ln 15)$. At the first one the function $f$ has a saddle while at the second one it has local minimum.

*Remark. One may also evaluate the second derivatives of $f$, $g$ and $h$ at the critical points and decide by investigating the $d^2f$, $d^2g$ and $d^2h$. We have*

$\frac{\partial f}{\partial x} = 30(x^4 + 2x^3 - 5x^2 - 6x) + x(x+3)(2x+3)((x^2+3x)^2 - e^{2y})$,

$\frac{\partial f}{\partial y} = -e^{2y}(-e^{2y} + (x^2+3x)^2)$ *and*

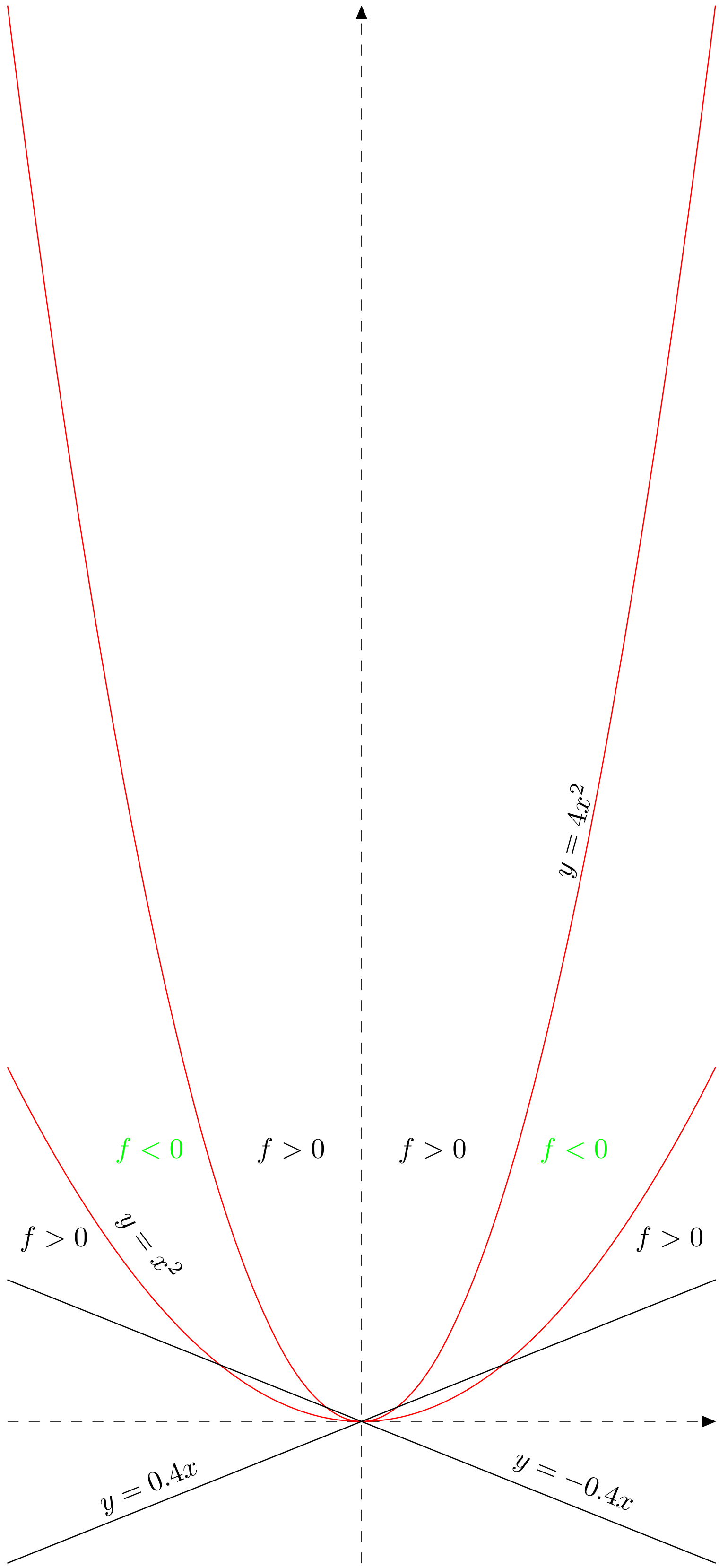
$\frac{\partial^2 f}{\partial x^2} = 60(2x^3 + 3x^2 - 5x - 3) + (6x^2 + 18x + 9)((x^2+3x)^2 - e^{2y}) + 2x^2(x+3)^2(2x+3)^2$,

$\frac{\partial^2 f}{\partial x\partial y} = -2x(x+3)(2x+3)e^{2y}$, $\quad \frac{\partial^2 f}{\partial y^2} = 4e^{4y} - 2e^{2y}x^2(x+3)^2) = 2e^{2y}\,(2e^{2y} - x^2(x+3)^2)$.

*Therefore* $d^2f(-1,\ln 2) = \begin{pmatrix} 188 & 16 \\ 16 & 32 \end{pmatrix}$ *and* $d^2f(2,\ln 10) = \begin{pmatrix} 10800 & -14000 \\ -14000 & 20000 \end{pmatrix}$. *The entries at left upper corners are positive, also the determinants of both matrices are positive so the matrices are positively defined and the function $f$ has local minima at both points.*

*There are no other critical points so there are no saddles nor local maxima. The function $f$is unbounded from above since $f(x,0) \xrightarrow[x\to\infty]{} \infty$. Since $f(x,\ln((x+1)(x+3))) \xrightarrow[x\to-\infty]{} -\infty$ it is also unbounded from below .*

*4.* Let $f(x,y) = (x^2 - y)(4x^2 - y)$. Prove that if the domain of $f$ is restricted to any line $L$ through the origin then it has a proper local minimum at $(0,0)$, i.e. there is $\delta > 0$ such that if $\mathbf{p} \in L$ and $\|\mathbf{p}\| < \delta$ and $\mathbf{p} \neq (0,0)$ then $f(\mathbf{p}) > f(0,0) = 0$. Prove that $f$ does not have local minimum at the origin.




*Solution.* We have $f(0,y) = y^2$ and obviously this function attains its smallest value at 0. We are done with one line. Let us assume now that $y = ax$ for some real number $a \neq 0$. We have
$f(x, ax) = (x^2 - ax)(4x^2 - ax) =$
$= x^2(a - x)(a - 4x)$. If $0 < |x| < \frac{|a|}{4}$ then $x^2(a-x)(a-4x) > 0$ because either both numbers $(a-x), (a-4x)$ are positive or both are negative so their product is positive. So $f$ restricted to the line $y = ax$ has local minimum at the point $(0,0)$. Since $f(x,0) = 4x^4$ the same is true for the $x$–axis. On the other hand $f(x, 2x^2) = -2x^4 < 0$ for all $x \neq 0$. Therefore at any neighbourhood of the origin one can find a point at which the value of the function is less (strictly) than at $(0,0)$. Therefore the function $f$ does not have minimum at $(0,0)$.

*Remark. The inequality $f(x,y) > 0$ holds if either $y > 4x^2$ or if $y < x^2$. The inequality $f(x,y) < 0$ holds if $4x^2 > y > x^2$. This in equality changes when one of the red curves is overstepped*