Some problems to be solved at home. Choose 2 of them, write down your solutions and mail them to me.

1. Let $f\colon U \longrightarrow \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^k$ differentiable at a point $p \in U$ (the definition of a differentiable function can be found in prof. Warhursts's note, page 6). Let us assume that $\nabla f(p) \neq (0, 0, \ldots, 0)$. Prove that if $\|\mathbf{v}\| = 1$ and $\alpha(t) = f(\mathbf{p} + t\mathbf{v})$ then $-\|\nabla f(\mathbf{p})\| \leqslant \alpha'(0) \leqslant \|\nabla f(\mathbf{p})\|$. For what $\mathbf{v}$ the equality holds on the right hand side?

Solution. Let $\mathbf{v} = (v_1, v_2, \ldots, v_k)$. By the chain rule we have $\alpha'(t) = \sum_{j=1}^k \frac{\partial f}{\partial x_j}(p + t\mathbf{v})v_j$, so $\alpha'(0) = \sum_{j=1}^k \frac{\partial f}{\partial x_j}(p)v_j$. By Cauchy–Schwarz inequality we have $|\alpha'(0)| = \left|\sum_{j=1}^k \frac{\partial f}{\partial x_j}(p)v_j\right| \leqslant \sqrt{\sum_{j=1}^k (\frac{\partial f}{\partial x_j}(p))^2} \cdot \sqrt{\sum_{j=1}^k v_j^2} = \sqrt{\sum_{j=1}^k (\frac{\partial f}{\partial x_j}(p))^2}$. The equality on the right-hand side holds iff there exist $t > 0$ such that $\mathbf{v} = t \cdot \nabla f(\mathbf{p})$. This means that the vectors $\mathbf{v}$ and $\nabla f(\mathbf{p})$ are parallel and point in the same direction. If the vectors are parallel but point in the opposite directions then there exist w number $t < 0$ such that $\mathbf{v} = t \cdot \nabla f(\mathbf{p})$, then the left–hand side inequality holds. □

2. (a) Let $f(x, y) = ax^2 + 2bxy + cy^2$ for all $(x, y) \in \mathbb{R}^2$ for some fixed numbers a, b, c. Prove that $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$ iff $a > 0$ and $ac > b^2$ iff $c > 0$ and $ac > b^2$.
(b) Let $g(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz$ for all $x, y, z) \in \mathbb{R}^3$ for some fixed numbers A, B, C, D, E, F. Prove that $g(x, y, z) > 0$ for all $(x, y, z) \neq (0, 0, 0)$ iff $A > 0$, $AB > D^2$ and $A(BC - F^2) - D(DC - EF) + E(DF - BE) > 0$.
High school algebra is sufficient for a solution of this problem.
One may use the result of the part (a) in a solution of the part (b).

Solution. (a) Let us assume at first that $f(x, y) > 0$ for all points $(x, y) \neq (0, 0)$. Then $0 < f(1, 0) = a$. We may write $f(x, y) = ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 - \frac{b^2y^2}{a} + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{ca-b^2}{a}y^2$. Since $0 < f(b, -a) = \frac{ca-b^2}{a}(-a)^2 = a(ca - b^2)$ we have $ac > b^2$.
Let us assume now that $a > 0$ and $ac > b^2$ We may write $f(x, y) = ax^2 + 2bxy + cy^2 = {=}a\left(x + \frac{b}{a}y\right)^2 + \frac{ca-b^2}{a}y^2$. Both summands are nonnegative so the sum is nonnegative. It may be 0 only if both summands vanish. If $\frac{ca-b^2}{a}y^2 = 0$ then $y = 0$ but then $a\left(x + \frac{b}{a}y\right)^2 = ax^2$ so it is 0 only when $x = 0$. QED

(b) Let us assume at first that $g(x, y, z) > 0$ for all points $(x, y, z) \neq (0, 0, 0)$, therefore $0 < g(1, 0, 0) = A$. We have

$$0 = g(x, y, z) = A\left(x + \tfrac{D}{A}y + \tfrac{E}{A}z\right)^2 + \left(B - \tfrac{D^2}{A}\right)y^2 + \left(C - \tfrac{E^2}{A}\right)z^2 + 2\left(F - \tfrac{DE}{A}\right)yz.$$

We have $0 < g\left(-\frac{D}{A}y - \frac{E}{A}z, y, z\right) = \left(B - \frac{D^2}{A}\right)y^2 + \left(C - \frac{E^2}{A}\right)z^2 + 2\left(F - \frac{DE}{A}\right)yz$. Therefore for each $(y, z) \neq (0, 0)$ the inequality $(AB - D^2)y^2 + (AC - E^2)z^2 + (AF - DE)yz > 0$ holds (note that$A > 0$). By part (a) we have $AB - D^2 > 0$ and $0 < (AB - D^2)(AC - E^2) - (AF - DE)^2 = = A^2BC - ACD^2 - ABE^2 - A^2F^2 + 2ADEF = A(ABC - CD^2 - BE^2 - AF^2 + 2DEF) = = A\big(A(BC - F^2) - D(CD - EF) + E(DF - BE)\big)$, so $A(BC - F^2) - D(CD - EF) + E(DF - BE) > 0$. The proof of the necessity is done.
Now we assume that $A > 0$, $AB > D^2$ and $A(BC - F^2) - D(CD - EF) + E(DF - BE) > 0$. By part (a) we know that $\left(B - \frac{D^2}{A}\right)y^2 + \left(C - \frac{E^2}{A}\right)z^2 + 2\left(F - \frac{DE}{A}\right)yz > 0$ for all $(y, z) \neq (0, 0)$. Therefore $g(x, y, z) = A\left(x + \frac{D}{A}y + \frac{E}{A}z\right)^2 + \left(B - \frac{D^2}{A}\right)y^2 + \left(C - \frac{E^2}{A}\right)z^2 + 2\left(F - \frac{DE}{A}\right)yz \geqslant 0$ for all $(x, y, z) \neq (0, 0, 0)$. Moreover if $g(x, y, z) = 0$ then $x + \frac{D}{A}y + \frac{E}{A}z = 0$ and $\left(B - \frac{D^2}{A}\right)y^2 + \left(C - \frac{E^2}{A}\right)z^2 + 2\left(F - \frac{DE}{A}\right)yz = 0$, so $y = 0 = z$ and thus $x = 0$. QED. □

Remark. The condition may be written as follows

$$A > 0, \qquad \begin{vmatrix} A & D \\ D & B \end{vmatrix} > 0, \qquad \begin{vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{vmatrix} > 0.$$

The similar theorem is true for functions of arbitrarily number of the variables, it is called Sylvester theorem.
One can also prove it using the discriminant of a quadratic polynomial

$$Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz=Ax^2+2(Dy+Ez)x+By^2+2Fyz+Cz^2$$

of the variable x with y,z kept constant. The discriminant equals

$$4(Dy+Ez)^2-4A(By^2+2Fyz+Cz^2)=4\left((D^2-AB)y^2+2(DE-AF)yz+(E^2-AC)z^2\right).$$

If all values are positive then the discriminant must be negative etc. This is equivalent to what was done above, recall the canonical form of quadratic polynomials. □

3. Let $F(r,\varphi)=(r\cos\varphi, r\sin\varphi)$ for $r>0$ and $\varphi\in\mathbb{R}$.
Let $\frac{\partial}{\partial r}F=\left(\frac{\partial}{\partial r}(r\cos\varphi),\frac{\partial}{\partial r}(r\sin\varphi)\right)$ and $\frac{\partial}{\partial\varphi}F=\left(\frac{\partial}{\partial\varphi}(r\cos\varphi),\frac{\partial}{\partial\varphi}(r\sin\varphi)\right)$.
Find the angle between the vectors $\frac{\partial}{\partial r}F$ and $\frac{\partial}{\partial\varphi}F$. How does the angle depend on (r,φ)?

Solution. We have $\frac{\partial}{\partial r}(r\cos\varphi)=\cos\varphi$ and $\frac{\partial}{\partial r}(r\sin\varphi)=\sin\varphi$ so $\frac{\partial}{\partial r}F=(\cos\varphi,\sin\varphi)$. Other equations are $\frac{\partial}{\partial\varphi}(r\cos\varphi)=-r\sin\varphi$, $\frac{\partial}{\partial\varphi}(r\sin\varphi)=r\cos\varphi$ thus $\frac{\partial}{\partial\varphi}F=\left(-r\sin\varphi, r\cos\varphi\right)$. Therefore
$\frac{\partial}{\partial r}F\cdot\frac{\partial}{\partial\varphi}F=\left\langle\frac{\partial}{\partial r}F,\frac{\partial}{\partial\varphi}F\right\rangle=\langle(\cos\varphi,\sin\varphi),(-r\sin\varphi,r\cos\varphi)\rangle=\cos\varphi(-r\sin\varphi)+\sin\varphi(r\cos\varphi)=0.$
This means that that the vectors are perpendicular. The angle made by them is 90° or $\frac{\pi}{2}$ radians independently of the point. □

4. Find all points of differentiability of the three functions defined on $\mathbb{R}^3$:

(a) $\mathbf{x}\mapsto\|\mathbf{x}\|_1$ (b) $\mathbf{x}\mapsto\|\mathbf{x}\|_2$, (c) $\mathbf{x}\mapsto\|\mathbf{x}\|_\infty$.

You may draw the sets defined by the equations $\|\boldsymbol{x}\|_1=1$, $\|\boldsymbol{x}\|_2=1$, $\|\boldsymbol{x}\|_\infty=1$. *This should help in guessing the answer.*

Solution. $\|(x,y,z)\|_1=|x|+|y|+|z|$, so $\frac{\partial}{\partial x}(\|(x,y,z)\|_1)=\frac{\partial}{\partial x}(|x|)=\frac{|x|}{x}$ for all (x,y,z) with $x\neq0$. Also $\frac{\partial}{\partial y}(\|(x,y,z)\|_1)=\frac{\partial}{\partial y}(|y|)=\frac{|y|}{y}$ for all (x,y,z) with $y\neq0$ and $\frac{\partial}{\partial z}(\|(x,y,z)\|_1)=\frac{\partial}{\partial z}(|z|)=\frac{|z|}{z}$ with $z\neq0$. These partial derivatives are continuous as functions of (x,y,z) at all points (x,y,z) with $xyz\neq0$. Hence $\|\cdot\|$ is differentiable at all such points. At points with $x=0$ the derivative $\frac{\partial}{\partial x}$ does not exist so norm is not differentiable at such points. The similar statements are true in the case of 2 other partial derivatives. The norm $\|\cdot\|_1$ is differentiable at a point (x,y,z) iff $xyz\neq0$.
$\frac{\partial}{\partial x}(\|(x,y,z)\|_2=\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot2x=\frac{x}{\sqrt{x^2+y^2+z^2}}$
if $x^2+y^2+z^2>0$. The function $(x,y,z)\mapsto\frac{x}{\sqrt{x^2+y^2+z^2}}$ is continuous at every point $(x,y,z)\neq(0,0,0)$. The same is true for functions $(x,y,z)\mapsto\frac{\partial}{\partial y}(\|(x,y,z)\|_2=\frac{y}{\sqrt{x^2+y^2+z^2}}$ and $(x,y,z)\mapsto\frac{\partial}{\partial z}(\|(x,y,z)\|_2=\frac{z}{\sqrt{x^2+y^2+z^2}}$ At the point $(0,0,0)$ the function is not differentiable because $\|(x,0,0)\|=|x|$ so partial derivative relative to x does not exist at the origin.
$\|(x,y,z)\|_\infty=\max(|x|,|y|,|z|)$. If $|x|\neq|y|\neq|z|\neq|x|$ then one of the three numbers is the largest. Let it be $|x|$. Then there exists a number $\delta>0$ such that if $|x-u|<\delta$ and $|y-v|<\delta$ and $|z-w|<\delta$ then $|u|>|v|$ and $|u|>|w|$ therefore $\|(u,v,w)\|_\infty=|u|\neq0$. Therefore $\|\cdot\|_\infty$ is differentiable around (x,y,z) and the gradient of it is $(\frac{|x|}{x},0,0)$ so it is either $(1,0,0)$ or $(-1,0,0)$. The problem with the differentiability arises if $|x|=|y|\geqslant|z|$ or $|y|=|z|\geqslant|x|$ or $|z|=|x|\geqslant|y|$. Suppose the first case holds and $x>0$. Then $\lim_{h\to0^+}\frac{\|(x+h,y,z)\|_\infty-\|(x,y,z)\|_\infty}{h}=1$ while $\lim_{h\to0^-}\frac{\|(x+h,y,z)\|_\infty-\|(x,y,z)\|_\infty}{h}=0$ because $\|(x+h,y,z)\|_\infty-\|(x,y,z)\|_\infty=x+h-x=h$ for $h>0$ and $\|(x+h,y,z)\|_\infty-\|(x,y,z)\|_\infty=|y|-|y|=0$ for $-x<h<0$. At $(0,0,0)$ the norm is not differentiable because $\|(x,0,0)\|_\infty=|x|$ so even partial derivative relative to x does not exist at this point. This ends the solution, □

Remark The equation $|x|=|y|$ describes the union of 2 planes in $\mathbb{R}^3$ as do equation $|x|=|z|$ and $|y|=|z|$. Outside of the union of these six planes the norm $\|\cdot\|$ is differentiable, at some points of these planes the norm differentiable, too. E.g. at $(2,2,3)$, in general at all points at which 2 coordinates are equal in the absolute value and the absolute value of the third one is greater (not equal but greater). There are six „infinite quadrilateral cones" with a common vertex at $(0,0,0)$ and their faces are contained in the union of the six planes described above.

5. Prove that among the triangles inscribed into the circle of radius 1 there is a triangle with the largest area.

Solution. Let (x_1,y_1), (x_2,y_2), (x_3,y_3) be the vertices of a triangle. Therefore the equalities $x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2=1$ hold so the set consisting of points $(x_1,y_1,x_2,y_2,x_3,y_3)\in\mathbb{R}^6$ is bounded and closed in $\mathbb{R}^6$ so it is compact. By Weierstrass Maximum Principle the function

$$A(x_1,y_1,x_2,y_2,x_3,y_3)=\tfrac{1}{2}\left|(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)\right|$$

which is continuous attains its largest value at some point of this set. The value of the function A is the area of the triangle with the vertices (x_1,y_1), (x_2,y_2), (x_3,y_3). Maximal value of A is positive because because it is not less than the area of the triangle with the vertices $(1,0)$, $(0,1)$ and $(-1,0)$ which is equal to 1. The minimal value of A is 0. This is obvious and happens when 2 vertices coincide.

6. Prove that among the 17-gons inscribed into the circle of radius 1 there is a 17-gon with the largest area. (17-gon has 17 vertices.)

Solution. We may say the vertices of the 17-gon are of the form $V_1=(\cos t_1,\sin t_1)$, $V_2=(\cos t_2,\sin t_2)$, $V_3=(\cos t_3,\sin t_3)$,..., $V_{17}=(\cos t_{17},\sin t_{17})$, with $0\leqslant t_1\leqslant t_2\leqslant t_3\leqslant\ldots\leqslant$ $\leqslant t_{17}\leqslant 2\pi$. The inequalities are NOT sharp so some vertices may coincide which means that a degenerate 17–gons may appear. We have to admit degenerate polygons to ensure compactness of the domain. The area of the triangle $V_1V_jV_{j+1}$, $j=2,3,4,\ldots,16$, is equal to

$$\tfrac{1}{2}\left|(\cos t_j-\cos t_1)(\sin t_{j+1}-\sin t_1)-(\sin t_j-\sin t_1)(\cos t_{j+1}-\cos t_1)\right|$$

so it is a continuous function of $(t_1,t_2,t_3,\ldots,t_{17})$. The sum of these 15 continuous function is also continuous thus the are of the 17-gon is a continuous function of $(t_1,t_2,t_3,\ldots,t_{17})$. Its domain is bounded and closed i.e. it is compact. Therefore By Weierstrass Maximum Principle it attains the maximal value. The only problem left is whether or not the obtained 17 points are really distinct, theoretically it could happen that e.g. two or three vertices coincide. This is not the case. Assume for example that $V_{j-1}\neq V_j=V_{j+1}$. Then one may replace V_j with the midpoint of the arc with ends V_{j-1},V_{j+1} and the new 17-gon will have greater area than the one which was supposed to have the greatest area. This contradiction shows that for the 17-gon with the biggest area all the vertices are distinct so this one is nondegenerate.

Remark. One can prove that the largest area has the regular 17-gon but you were not asked to do it.

7. Prove that among the flat cuts (cross-sections) of the cube with edge of length 1 there is one with the largest area.

Remark. It is **not** true that among the pentagons which are cross-sections of the given cube there is one with the largest area.

The solution appears at another file that I have mailed to all students of group 2.