Some problems to be solved at home. Choose 2 of them, write down your solutions and mail them to me.

1. Let $A = \{(x, y) \in \mathbb{R}^2 : \quad \exists_{r,s} x = e^r \cos s, y = e^r \sin s\}$ Is the set A
(a) closed; (b) open; (c) bounded; (d) compact; (e) convex (f) connected?
Please justify your answer.

Solution. We have $(e^r \cos s)^2 + (e^r \sin s)^2 = e^{2r}(\cos^2 s + \sin^2 s) = e^{2r} > 0$ so the distance from the point $(e^r \cos s, e^r \sin s)$ to the origin, i.e. to $(0, 0)$ is positive. Therefore $(0, 0) \notin A$. All other points (x, y) are in A. It is enough to define $r = \frac{1}{2}\ln(x^2 + y^2)$, the $e^{2r} = x^2 + y^2$. Then define s as such number that $\cos s = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin s = \frac{y}{\sqrt{x^2+y^2}}$. Such s exists because each point of the unite circle (centered at the origin of radius 1) can be written in the form $(\cos s, \sin s)$. We proved that $A = \mathbb{R}^2 \setminus \{(0, 0)\}$. This set is not closed because $(\frac{1}{n}, \frac{1}{n}) \in A$ and $\lim\limits_{n\to\infty} (\frac{1}{n}, \frac{1}{n}) \notin A$; it is open because its complement is one point set which is closed; it is unbounded because the distance from $(1, 0)$ to $(n, 0)$ equals n so it is as large as we want to for sufficiently big n; it is not convex because the straight line segment connecting $(1, 0)$ with $(-1, 0)$ contains a point not in A (the origin); it is connected because one can connect any two points $\mathbf{p}, \mathbf{q} \in A$ with a path consisting at most of two straight line segments: if $(0, 0)$ does not lie on the segment with ends $\mathbf{p}, \mathbf{q}$ then one segment suffices, if $(0, 0)$ lies on the segment with ends $\mathbf{p}, \mathbf{q}$ then we choose a point $\mathbf{r} \neq (0, 0)$ on the line perpendicular to the segment with end $\mathbf{p}, \mathbf{q}$ and take the path consisting of the segments joining $\mathbf{p}$ with $\mathbf{r}$ and $\mathbf{r}$ with $\mathbf{q}$. QED

2. Does there exist the limit $\lim_{(r,s)\to(\infty,\infty)} \big(e^r \cos s, e^r \sin s\big)$? If it does evaluate it, if it does not prove it.

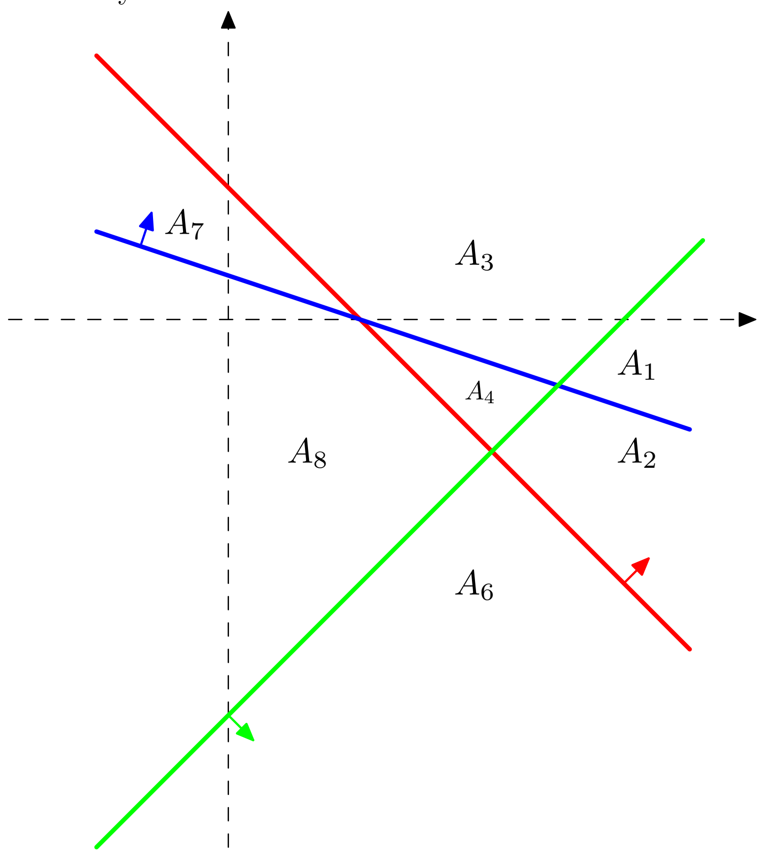
Solution. It does not. Let $r_n = n$, $s_n = \frac{3\pi}{2} + 2n\pi$ for $n = 1, 2, \ldots$ Then $\lim\limits_{n\to\infty} r_n = \infty = \lim\limits_{n\to\infty} s_n$ and $\lim\limits_{n\to\infty} e^{r_n}\big(\cos(s_n), \sin(s_n)\big) = \lim\limits_{n\to\infty} e^n\big(\cos(\frac{3\pi}{2}), \sin(\frac{3\pi}{2})\big) = \lim\limits_{n\to\infty} e^n(0, -1) = (0, -\infty)$.
On the other hand if $r_n = n$ and $s_n = 2n\pi$ for $n = 1, 2, 3, \ldots$ Then $\lim\limits_{n\to\infty} r_n = \infty = \lim\limits_{n\to\infty} s_n$ and $\lim\limits_{n\to\infty} e^{r_n}\big(\cos(s_n), \sin(s_n)\big) = \lim\limits_{n\to\infty} e^n\big(\cos(2n\pi), \sin(2n\pi)\big) = \lim\limits_{n\to\infty} e^n(1, 1) = (\infty, 0)$.
The limits are distinct, so the limit in question does not exist. QED.

3. Does there exist the limit $\lim_{(r,s)\to(0^+,0^+)} r^s$? We assume in this problem that the function $(r, s) \mapsto r^s$ is defined in the first quadrant $Q_{++} = \{(r, s) : \quad r > 0, s > 0\}$.
Justify your answer carefully.

Solution. It does not. Let $r_n = (\frac{1}{2})^n$, $s_n = \frac{1}{n}$. Obviously $r_n^{s_n} = \frac{1}{2}$ so $\lim\limits_{n\to\infty} r_n^{s_n} = \frac{1}{2}$. If the limit exists it must be $\frac{1}{2}$. But we can set $r_n = (\frac{2}{3})^n$ and $s_n = \frac{1}{n}$. Then $\lim\limits_{n\to\infty} r_n^{s_n} = \frac{2}{3} \neq \frac{1}{2}$ QED.
Remark. We can obtain as this limit any number from the interval $[0, 1]$, e.g. if $r_n = (\frac{1}{n})^n$, $s_n = \frac{1}{n}$ then Then $\lim\limits_{n\to\infty} r_n^{s_n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

4. Let
$A_1 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \geqslant 1,\ x - y \geqslant 3,\ x + 3y \geqslant 1\}$,
$A_2 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \geqslant 1,\ x - y \geqslant 3,\ x + 3y \leqslant 1\}$,
$A_3 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \geqslant 1,\ x - y \leqslant 3,\ x + 3y \geqslant 1\}$,
$A_4 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \geqslant 1,\ x - y \leqslant 3,\ x + 3y \leqslant 1\}$
$A_5 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \leqslant 1,\ x - y \geqslant 3,\ x + 3y \geqslant 1\}$,
$A_6 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \leqslant 1,\ x - y \geqslant 3,\ x + 3y \leqslant 1\}$,
$A_7 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \leqslant 1,\ x - y \leqslant 3,\ x + 3y \geqslant 1\}$,
$A_8 = \{(x, y) \in \mathbb{R}^2 : \quad x + y \leqslant 1,\ x - y \leqslant 3,\ x + 3y \leqslant 1\}$.
Draw the sets on the plane. Decide in each case whether or not the set is
(a) closed; (b) open; (c) bounded; (d) compact; (e) convex (f) connected?
Please justify your answer.
The problem is much easier than you could expect judging by its length.

Solution. Equations All 8 sets are closed because they are defined by non-strict inequalities amon continuous functions. $x+y=1$, $x-y=3$ and $x+3y=1$ describe the lines perpendicular to the vectors $[1,1]$, $[1,-1]$ and $[1,3]$. The first 2 meet at point $(2,-1)$, the first and the third meet at the point $(1,0)$, the second and the third meet at the point $(\frac{5}{2},-\frac{1}{2})$. They divide the plane into seven regions one is bounded. A_4 is a triangle with the vertices $(1,0)$, $(2,_1)$ and $(\frac{5}{2},-\frac{1}{2})$. A_5 is empty (it contains no point) because if $3\leqslant x-y$ and $1\leqslant x+3y$ then $4\leqslant 2x+2y$ so $2\leqslant x+y$ contrary to $x+y\leqslant 1$. These two sets are bounded and closed so they are compact. Also they are convex and therefore connected.



Coloured short arrows show the direction in which the appropriate function grows the fastest.
Equation of the red line:
$x+y=1$,
equation of the blue line:
$x+3y=1$,
equation of the green line:
$x-y=34$.

The triangle is not open but the empty set is open. All 6 other sets are unbounded for each of them contains a half line, they are convex as intersections of convex sets (half planes) and therefore connected. The vectors mentioned above show in which direction each of the functions $x+y$, $x-y$ and $x+3y$ grows the fastest, these are gradients of the functions. This allows to draw the picture of seven non-empty regions and quickly give them correct names – quite a few people did that. The best thing was to draw one picture for all sets. The end of the problem 4.

5. Let the set G consists of such $(x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}^6$ that three points (x_1,x_2), (x_3,x_4), (x_5,x_6) do not lie on one straight line, in particular no two of them coincide. Let H be the common point of the altitudes of the triangle with vertices (x_1,x_2), (x_3,x_4), (x_5,x_6). Is the set G open in $\mathbb{R}^6$? Is it compact? Is the map $(x_1,x_2,x_3,x_4,x_5,x_6)\mapsto H$ a continuous function mapping G into $\mathbb{R}^2$?
In this problem an altitude of a triangle is a straight line through one of the vertices perpendicular to the line through two other vertices.

Solution. Nobody decided to solve this problem. Maybe you regarded this as too easy for clever students. The line containing the altitude from (x_1,x_2) i.e. perpendicular to the line through (x_3,x_4) and (x_5,x_6) may be described by the equation $(x_3-x_5)x+(x_4-x_6)y=(x_3-x_5)x1+(x_4-x_6)x_2$. This so because the vector $[A,B]$ is perpendicular to the line $Ax+By+C=0$. The line containing the altitude from(x_3,x_4) i.e. perpendicular to the line through (x_1,x_2) and (x_5,x_6) may be described by the equation $(x_1-x_5)x+(x_2-x_6)y=(x_1-x_5)x3+(x_2-x_6)x_4$. If $H=(x,y)$ the the numbers x,y satisfy both linear equations and this is the unique solution of this system. Therefore x and y are quotients of polynomial expressions in variables x_1,x_2,x_3,x_4,x_5,x_6 so by theorems from prof. Warhurst's notes they are continuous. We do need the formulas. their form is sufficient for establishing the continuity. QED

Remark. It is not hard to do the unnecessary job i.e. to give explicit formulas for x and y. Namely

$$x = \frac{((x_3-x_5)x_1+(x_4-x_6)x_2)(x_2-x_6)-((x_1-x_5)x_3+(x_2-x_6)x_4)(x_4-x_6)}{(x_3-x_5)(x_2-x_6)-(x_4-x_6)(x_1-x_5)}$$

and

$$y = \frac{((x_1-x_5)x_3+(x_2-x_6)x_4)(x_3-x_5)-((x_3-x_5)x_1+(x_4-x_6)x_2)(x_1-x_5)}{(x_3-x_5)(x_2-x_6)-(x_4-x_6)(x_1-x_5)}.$$

The denominator is different from 0 because the triangle exists, on can prove that the absolute value of it is doubled area of the triangle under consideration.

The distance from the point (x_1,x_2) to the line through (x_3,x_4) and (x_5,x_6) described by the equation $x(x_6-x_4)+y(x_3-x_5)-x_3(x_6-x_4)-x_4(x_3-x_5)$ equals

$$\frac{|x_1(x_6-x_4)+x_2(x_3-x_5)-x_3(x_6-x_4)-x_4(x_3-x_5)|}{\sqrt{(x_6-x_4)^2+(x_3-x_5)^2}} =$$

$$= \frac{|(x_1-x_3)(x_6-x_4)+(x_2-x_4)(x_3-x_5)|}{\sqrt{(x_6-x_4)^2+(x_3-x_5)^2}}.$$

Therefore the area of the triangle with the vertices $(x_1,x_2),\ (x_3,x_4),\ (x_5,x_6)$ equals to

$$\frac{1}{2}\frac{|(x_1-x_3)(x_6-x_4)+(x_2-x_4)(x_3-x_5)|}{\sqrt{(x_6-x_4)^2+(x_3-x_5)^2}}\cdot\sqrt{(x_6-x_4)^2+(x_3-x_5)^2} =$$

$$= \frac{1}{2}|(x_1-x_3)(x_6-x_4)+(x_2-x_4)(x_3-x_5)|$$

It is easy to see that

$|(x_1-x_3)(x_6-x_4)+(x_2-x_4)(x_3-x_5)| = |(x_3-x_5)(x_2-x_6)-(x_4-x_6)(x_1-x_5)| =$
$= |(x_5-x_1)(x_4-x_2)-(x_6-x_2)(x_3-x_1)| = |x_1x_6-x_1x_4+x_2x_3-x_2x_5-x_3x_6+x_4x_5$. This would look better if determinants have been used but I am not sure you are familiar with them.

Let me write it anyway. The doubled area of the triangle with the vertices $(x_1,x_2),\ (x_3,x_4),\ (x_5,x_6)$ equals to the absolute value of the determinant

$$\begin{vmatrix} x_1-x_3 & x_2-x_4 \\ x_5-x_3 & x_6-x_4 \end{vmatrix} = \begin{vmatrix} x_3-x_5 & x_4-x_6 \\ x_1-x_5 & x_2-x_6 \end{vmatrix} = \begin{vmatrix} x_5-x_1 & x_6-x_2 \\ x_3-x_1 & x_4-x_2 \end{vmatrix} = -\begin{vmatrix} 1 & x_1 & x_2 \\ 1 & x_3 & x_4 \\ 1 & x_5 & x_6 \end{vmatrix}$$