Definition 0.1 (of an open set)
A set $G \subseteq \mathbb{R}^k$ is open iff for each point $\mathbf{p} \in G$ there exist a number $r_{\mathbf{p}} > 0$ such that if $\mathbf{x} \in \mathbb{R}^k$ and $\|\mathbf{x} - \mathbf{p}\|_2 < r_{\mathbf{p}}$ then $\mathbf{x} \in G$.

It means that if $\mathbf{p} \in G$ and $\mathbf{x}$ is a point in $\mathbb{R}^k$ which is sufficiently close to $\mathbf{p}$ then $\mathbf{x}$ also is in G. The result depends on k. Let us look at few examples.

Example 0.2 $k = 1$, $G = (3, 7)$. If $p \in (3, 7)$ and $r_p = min(|3 - p|, |7 - p|)$ that is r_p is the smaller of the 2 numbers $|a - p|$ and $|b - p|$ and if $|x - p| < r_p$ then $x \in (a, b)$ because the distance from p to x is less than the distance from p to any end of the interval (a, b). So open intervals are open sets in $\mathbb{R}$ or we may say any open interval is an open subset of $\mathbb{R}$. $\square$

Example 0.3 $k = 1$, $G = (3, 7]$. This set is **not** open. The problem is with $p = 7$ We can find always a number close to 7 which is outside of $G = (3, 7]$. For example the distance from $7 + \frac{r}{2}$ is $\frac{r}{2} < r$ and $7 + \frac{r}{2} \notin (3, 7]$.

Example 0.4 $k = 1$, $G = \{x \in \mathbb{R} \colon \quad x \neq 0\}$. The set G is open. The reason is that if $p \in G$, so $p \neq 0$ and if $r = |p|$ and $|x - p| < r$ then $x \neq 0$. Formally we may write: if $|p| - |x| \leqslant |p - x| < r = |p|$ then $0 < |x|$, the first inequality is the triangle inequality, then it is simplified. $0 < |x|$ implies that $x \neq 0$ so $x \in G$.

Proposition 0.5 The set $G \subset \mathbb{R}$ is open if and only if it is a union of arbitrary number of open intervals.

This proposition is quite obvious. If the set G is open then by the definition of the open set then together with a point p it contains the set $(p - r_p, p + r_p)$ for some $r_p > 0$. If the set G is a union of open intervals then each point of G is at at least one these open intervals and we can argue as in the example 0.2. $\square$

Example 0.6 $k = 2$, $G = \{(x, y) \colon \quad x^2 + (y - 2)^2 < 9$ is an open set. Clearly G is a disc of radius 3 centered at $(0, 2)$ without a circle of radius 3 centered at $(0, 2)$, i.e. without boundary. Let us prove that G is an open set. Let $(p_1, p_2) = \mathbf{p} \in G$. This means that $p_1^2 + (p_2 - 2)^2 < 9$. Let $r_{\mathbf{p}} = 3 - \sqrt{p_1^2 + (p_2 - 2)^2}$. We are going to prove that the inequality $(x - p_1)^2 + (y - p_2)^2 < r_p^2$ implies that $G = \{(x, y) \colon \quad x^2 + (y - 2)^2 < 9$ so we are going to prove that the disc of radius r_p centered at $\mathbf{p}$ is contained in the disc of radius 3 centered at $(0, 2)$. This is an immediate consequence of the triangle inequality

$$\|(x, y) - (0, 2)\|_2 \leqslant \|(x, y) - \mathbf{p}\|_2 + \|\mathbf{p} - (0, 2)\|_2 < (3 - r_{\mathbf{p}}) + r_{\mathbf{p}} = 3. \ \square$$

Remark 0.7 One can prove in the same way that the set $\{\mathbf{x} \in \mathbb{R}^k \colon \quad \|\mathbf{x} - \mathbf{p}\|_2 < r\}$ is an open subset of $\mathbb{R}^k$. $\square$

Example 0.8 The set $G = \{(x, 0) \colon \quad 0 < x < 1\}$ is NOT an open subset of the plane, so the open straight line segment is not open subset of the plane although it is an open subset of the x–axis. It is so because if $0 < y < r$ then $\|(x, y) - (x, 0)\|_2 = y < r$ and $(x, y) \notin G$. We just showed that the notion of open set depends on the space containing it. $\square$

Remark 0.9 If M is an arbitrary set in which the distance d between any two points is defined the we can talk of open sets in it. The distance d is a non-negative function which assigns a non-negative number $d(\mathbf{x},\mathbf{y})$ to the pair of points $\mathbf{x},\mathbf{y}\in M$so that
(i) $d(\mathbf{x},\mathbf{y})=0$ if and only if $\mathbf{x}=\mathbf{y}$ for all $\mathbf{x},\mathbf{y}\in M$;
(ii) $d(\mathbf{x},\mathbf{y})=d(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y}\in M$;
(iii) $d(\mathbf{x},\mathbf{z})\leqslant d(\mathbf{x},\mathbf{y})+d(\mathbf{y},\mathbf{z})$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in M$.
If such distance (or metric) d is defined in M we say that M is a metric space. The distance may de defined in different ways. One may say that the distance from the north pole to the south pole is around 12 742 kilometers but if one wanted to travel from one place to the second one then we would say that the distance is around 20020 kilometers. Also the symmetry condition we assume is not very obvious. If one is hiking in mountains it is usually so that if one walks up then he needs more time then for walking down. Frequently the distance in mountains is measured by the time needed for a walk. Nonetheless we assume symmetry of the distance. In any metric space we can define open sets. If the set is contained indifferent metric space it may may be open in only one of them. So when we talk of open sets it is necessary to indicate a metric space to which the notion is referred. □

A closed set $F\subset\mathbb{R}^k$ is such set that $\mathbb{R}^k\setminus F$ is open (in $\mathbb{R}^k$). A closed interval $[a,b]$ is a closed subset of $\mathbb{R}$. The set $\{(x,y)\in\mathbb{R}^2:\quad x^2+(y-2)^2\leqslant 3\}$ is a closed subset of $\mathbb{R}^2$. The set $[13,\infty)$ is a closed subset of $\mathbb{R}^2$. A graph of a continuous function defined on a closed interval is a closed subset of $\mathbb{R}^2$.

Proposition 0.10 The set $F\subset M$ is closed in M if and only if it follows from the equation $\mathbf{p}=\lim\limits_{n\to\infty}\mathbf{p}_n$ and from the sentence $\forall_n \mathbf{p}_n\in F$ that $\mathbf{p}\in F$. □
This may be read: the limit of the sequence with terms in F is in F.

By this proposition the interval $I=(1,100]$ is not closed. For each n the point $\frac{n+1}{n}$ is I while $\lim\limits_{n\to\infty}\frac{n+1}{n}=1\notin I$.

$\emptyset$ and $\mathbb{R}^k$ are the only subsets of $\mathbb{R}^k$ which are at the same time open and closed. A proof if the theorem saying that no other subset of $\mathbb{R}^k$ is open and closed is not trivial. We are not going to prove it.

The intersection of finitely many open subsets of $\mathbb{R}^k$ is necessarily open in $\mathbb{R}^k$. This sentence is equivalent to the following the union of finitely many closed subsets of $\mathbb{R}^k$ is a closed subset of $\mathbb{R}^k$.

The union of arbitrarily many of open subsets of $\mathbb{R}^k$ is open in $\mathbb{R}^k$. Equivalently the intersection of arbitrarily many of closed subsets of $\mathbb{R}^k$ is closed in $\mathbb{R}^k$.

Example 0.11 For each natural number n the interval $(-\frac{1}{n},\frac{1}{n})$ is open. $\bigcap\limits_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n})=\{0\}$ so the intersection of infinitely many open sets need not to be open. □

Example 0.12 Let $G_n=\{x\in\mathbb{R}:\quad x\neq n\}$ for $n=1,2,3,\ldots$. It is easy to see that G_n is open in $\mathbb{R}$ for each $n\in\mathbb{N}$. The set $G[=\bigcap\limits_{n=1}^{\infty}G_n$ consists of all real numbers which are not natural.

$x \in G = \bigcap_{n=1}^{\infty} G_n$ if and only if x is a real number different from $1, 2, 3, \ldots$. The set G is open. So it may happen that the intersection of infinitely many open sets may be open. $\square$

We may say that if a set in $\mathbb{R}^k$ is defined with finitely many strict inequalities between continuous functions then this set is open. We do not say this is a theorem. To make it precise we should say something about the domains of the functions in questions. But in simple situations it is true.

If in this sentence strict is replaced with $\leqslant$ or $\geqslant$ then the set is closed.

Example 0.13 The set $A = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leqslant 4,\ x+2y < 2\}$ is neither closed nor open. We are going to prove this. $(0, \frac{n}{n+1}) \in A$ for each natural number n because $0^2 + \left(\frac{n}{n+1}\right)^2 < 1 < 4$ and $0 + 2\frac{n}{n+1}) = \frac{2n}{n+1} < 2$. But $\lim\limits_{n\to\infty} (0, \frac{n}{n+1}) = (0,1)$ and it is not true that $0 + 2 \cdot 1 < 2$ hence $(0,1) \notin A$. By proposition 0.10 A is NOT open in $\mathbb{R}^2$. It is easy to see that $(-2,0) \in A$. But the distance from $(-2,0)$ to $(-2-\frac{r}{2}, 0)$ is $\frac{r}{2} < r$ and $(-2-\frac{r}{2}, 0) \notin A$ because $(-2-\frac{r}{2})^2 + 0^2 = 4 + 2r + r^2 > 4$ for any number $r > 0$. This proves that the set A is not open in $\mathbb{R}^2$. $\square$

Example 0.14 The set $A = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leqslant 1,\ x+y < 2\}$ is closed. We are going to prove it. If $x^2+y^2 \leqslant 1$ then $2 \geqslant 2x^2 + 2y^2 \geqslant x^2 + y^2 + 2xy = (x+y)^2$ thus $\sqrt{2} \geqslant |x+y|$, so $|x+y| \leqslant \sqrt{2} < 2$. This proves that

$$A = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leqslant 1,\ x+y < 2\} = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 \leqslant 1\}.$$

This proves that the set A is closed. $\square$

Example 0.15 The set $A = \{(x,y) : \quad x^2 < y < 4y^2\}$ is open in $\mathbb{R}^2$. Let us assume that $\mathbf{p} = (p_1, p_2) \in A$ that is $p_1^2 < p_2 < 4p_2^2$ and that $0 < \varepsilon < p_2 - p_1^2$ and $0 < \varepsilon < 4p_1^2 - p_2$. Then if $0 < \delta < 1$ and $\delta < \frac{\varepsilon}{3(2|p_1|+1)}$ and $|x - p_1| < \delta$ and $|y - p_2| < \delta$ then

$$x^2 \leqslant (|p_1| + \delta)^2 = p_1^2 + 2|p_1|\delta + \delta^2 < p_1^2 + 2|p_1|\delta + \delta = p_1^2 + (2|p_1|+1)\delta < p_1^2 + \tfrac{\varepsilon}{3}.$$

Thus if $|x - p_1| < \delta$ and $|y - p_2| < \frac{\varepsilon}{3}$ then $x^2 < p_1^2 + \frac{\varepsilon}{3} < p_2 - \frac{\varepsilon}{3} < y$. Let us assume that one more inequality is satisfied $\delta < \frac{\varepsilon}{24|p_1|}$. If $|x - p_1| < \delta$ then

$$4x^2 > 4(|p_1| - \delta)^2 = 4|p_1|^2 - 8|p_1|\delta + \delta^2 > 4|p_1|^2 - 8|p_1|\delta > 4|p_1|^2 - \tfrac{\varepsilon}{3}.$$

Thus if $|x - p_1| < \delta$ and $|y - p_2| < \frac{\varepsilon}{3}$ then $4x^2 > 4|p_1|^2 - \frac{\varepsilon}{3} > p_2 + \frac{\varepsilon}{3} > y$.
We proved that $\varepsilon > 0$ and $\delta > 0$ are chosen so that few inequalities we called upon above are satisfied then $x^2 < y < 4y^2$ so $(x,y) \in A$. This ends the prove. $\square$

The example above may be shortened if instead of finding specific estimates we decide to use continuity of the functions x^2, $4x^2$ and y. Then we just say that if $|x - p_1| < \delta$ the $4|x^2 - p_1^2| < \frac{\varepsilon}{3}$ where $\varepsilon > 0$ is a number such that $0 < \varepsilon < p_2 - p_1^2$ and $0 < \varepsilon < 4p_1^2 - p_2$. The existence of δ follows from the continuity of the functions x^2, $4x^2$. Specific value of δ is not needed for the proof.