8. Determine the maximum and minimum values of the function $f(x;y;z)$ on the set $S \subset \mathbb{R}^3$ where:

b) $f(x,y,z) = x^2+y^2+z^2$ and $S = \{(x,y,z)\colon \quad 3x+2y+z=6\}$.

Solution 1. The set S is a plane. The vector $(1,2,3)$ is orthogonal (perpendicular) to this plane. Therefore The set $L = \{(3t,2t,t)\colon \quad t\in\mathbb{R}$ is a straight line perpendicular to S. The line L contains the origin. It meets the plane S at the point $(3t,2t,t)$ for t such that $9t+4t+t=6$ i.e. for $t=\frac{6}{14}=\frac{3}{7}$. The common point of S and L is therefore $(\frac{9}{7},\frac{6}{7},\frac{3}{7})$. This means that the least value of f is $(\frac{9}{7})^2+(\frac{6}{7})^2+(\frac{3}{7})^2=\frac{81+36+9}{49}=\frac{18}{7}$. The function f is unbounded from above because if contains all points of the form $(0,t,6-2t)$ and $f(0,t,6-2t)=t^2+(6-2t)^2\geqslant t^2 \xrightarrow[t\to\infty]{} \infty$.

Solution 2. An extremal value of f may be attained at a point (x,y,z) at which the following equation $2(x,y,z)=\nabla f=\lambda(3,2,1)=\nabla(3x+2y+z-6)$ is satisfied for some number λ. Since $3x+2y+z=6$ the inequality $\lambda\neq 0$ holds. Therefore $x=3z$ and $y=2z$. This implies that $9z+4z+z-6=0$ so $z=\frac{3}{7}$, $y=\frac{6}{7}$ and $x=\frac{9}{7}$. This proves that **if** f attains its least upper bound or its greatest lower bound then it does this at a point $\left(\frac{9}{7},\frac{6}{7},\frac{3}{7}\right)$ so the bound is $f\left(\frac{9}{7},\frac{6}{7},\frac{3}{7}\right)=\frac{81+36+9}{49}=\frac{18}{7}$. At the moment we do not know whether $\frac{18}{7}$ is a bound of F and if it is whether $\sup f=\frac{18}{7}$ or $\inf f=\frac{18}{7}$. Let $\hat{S}=\{(x,y,z)\colon \quad 3x+2y+z=6 \quad \text{and} \quad x^2+y^2+z^2\leqslant 100\}$. Obviously $\hat{S}\subset S$ and if $(x,y,z)\in S\setminus\hat{S}$ then $f(x,y,z)=x^2+y^2+z^2>100>3>\frac{18}{7}$. The set $\hat{S}$ is compact thus the continuous function f attains its greatest lower bound at some point of $\hat{S}$. Obviously $\inf_S f=\inf_{\hat{S}} f<3$. Therefore $\inf_S f=\frac{18}{7}$. As in the first solution we prove that $\sup_S=\infty$. The solution is now complete. □

f) $f(x,y,z)=x^2+y^2+z^2 \quad S=\{(x,y,z)\in\mathbb{R}^3\colon \quad x+y+z=1,\ x+2y+3z=6\}$.

Solution 1. The set S the intersection of the 2 planes which are not parallel. Therefore S is a straight line. The function f is the square of the distance from (x,y,z) to $(0,0,0)$ so it is not bounded from above: $\sup_S f=\infty$. From the equations $x+y+z=1$ and $x+2y+3z=6$ it follows that $z-x=4$ so $z=4+x$ and $y=-3-2x$. This implies that $f(x,y,z)=x^2+(-3-2x)^2+(4+x)^2=6x^2+20x+25=6(x+\frac{20}{12})^2-6(\frac{5}{3})^2+25$. This implies that the smallest value of f is $-6(\frac{5}{3})^2+25=\frac{1}{3}(3\cdot 25-2\cdot 25)=\frac{25}{3}$. This may be written in the form $\inf_S f=\frac{25}{3}=f(-\frac{5}{3},\frac{1}{3},\frac{7}{3})=\frac{25+1+49}{9}=\frac{25}{3}$.

Solution 2. The gradients $\nabla(x+y+z-1)=(1,1,1)$ and $\nabla(x+2y+3z-6)=(1,2,3)$ are linearly independent. Therefore for every point (x,y,z) at which an extreme value is attained there exist numbers λ_1,λ_2 such that

$$\nabla(x^2+y^2+z^2)=2(x,y,z)=\lambda_1(1,1,1)+\lambda_2(1,2,3)=\lambda_1\nabla(x+y+z-1)+\lambda_1\nabla(x+2y+3z-6)$$

Therefore $2x=\lambda_1+\lambda_2$, $2y=\lambda_1+2\lambda_2$ and $2z=\lambda_1+3\lambda_2$. Now subtract the consecutive equations to get $2(y-x)=\lambda_2$ and $2(z-y)=\lambda_2$. Therefore $z-y=y-x$ i.e. $x-2y+z=0$. This implies that $3y=(x+y+z)-(x-2y+z)=1$ so $y=\frac{1}{3}$. Thus $x+z=\frac{2}{3}$ and $x+3z=\frac{16}{3}$ and $2z=\frac{16}{3}-\frac{2}{3}$ hence $z=\frac{7}{3}$ and $x=-\frac{5}{3}$. We proved that **if** an extreme value exists it is attained at the point $(-\frac{5}{3},\frac{1}{3},\frac{7}{3})$ so it equals $(-\frac{5}{3})^2+(\frac{1}{3})^2+(\frac{7}{3})^2=\frac{25}{3}$. As many times before let us restrict the domain of f. Let $\tilde{S}=\{(x,y,z)\in\mathbb{R}^3\colon \quad x+y+z=1,\ x+2y+3z=6, x^2+y^2+z^2\leqslant 100\}$. The set $\tilde{S}$ is bounded and closed i.e. it is compact. Therefore f attains its greatest lower bound which is not greater than $\frac{25}{3}$ because $(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3}) \in \tilde{S}$. But the point $(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3})$ is the only one at which extreme value can be attained so $\inf_{\tilde{S}} f = \frac{25}{3}$. The biggest value of f on $\tilde{S}$ os also attained but the point lying on the boundary of the ball $x^2 + y^2 + z^2 \leqslant 100$ and if we change the radius of the ball the maximum will change, too. We still need to notice that $\sup_S f = \infty$. This can be done either by saying that S is a straight line so the distance from the points of the line to the origin is unbounded from above or by indicating points from S which „escape" to ∞ e.g. $(x, -3 - 2x, 4 + x)$. We are done. □

9. Using the Kuhn-Tucker theorem, find the maximum value of the function $f(x, y) = x + ay$ on the set $M = \{(x, y) \in \mathbb{R}^2 \colon \quad x^2 + y^2 \leqslant 1, \ x + y \geqslant 0\}$ (the coefficient α is a fixed parameter).

Solution. The set M is half of the unit disk (with radius 1 centered at $(0, 0)$) above the line $x + y = 0$. the gradient $\nabla(x + y) = (1, 1)$ tells us in what direction the function $x + y$ grows the fastest. It is bounded by the semicircle from one side (NE) and by a straight line segments from another side (SW). The gradient of the optimized function $\nabla(x + \alpha y) = (1, \alpha)$ never vanishes. Due to this maximal and minimal values of f on M are attained at some boundary points of the set M which is compact that guarantees the existence of the extreme values. If (x, y) is a point at which one of them is attained the either $x + y = 0$ or $x^2 + y^2 = 1$. In the first case we have $f(x, y) = x + \alpha y = (-1 + \alpha)y$. This is a linear function in y. It attains its maximal and minimal values at the end of the interval $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ so they are $-\frac{-1+\alpha}{\sqrt{2}} = f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $\frac{-1+\alpha}{\sqrt{2}} = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ depending on its relation of α to 1. In the second case there exists λ such that $(1, \alpha) = \lambda(2x, 2y)$. If it does then $y = \alpha x$ and therefore $1 = x^2 + y^2 = x^2(1 + \alpha^2)$ thus $x = \pm\frac{1}{\sqrt{1+\alpha^2}}$ and $y = \alpha x = \pm\frac{\alpha}{\sqrt{1+\alpha^2}}$. The two obtained points or one of them should belong to M so $0 \leqslant x + y = \pm\frac{1+\alpha}{\sqrt{1+\alpha^2}}$. This tells us that if $\alpha \geqslant -1$ then $x + y = \frac{1+\alpha}{\sqrt{1+\alpha^2}} \geqslant 0$ and if $\alpha < -1$ then $x + y = -\frac{1+\alpha}{\sqrt{1+\alpha^2}} > 0$. $f\left(\frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}}\right) = \sqrt{1 + \alpha^2}$ and $f\left(-\frac{1}{\sqrt{1+\alpha^2}}, -\frac{\alpha}{\sqrt{1+\alpha^2}}\right) = -\sqrt{1 + \alpha^2}$. It is time to write the answer. There will be few cases.

Let $\alpha \geqslant -1$. In this case the points at which the extreme values can be attained are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\left(\frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}}\right)$. The corresponding values of f are: $\frac{1-\alpha}{\sqrt{2}}$, $\frac{\alpha-1}{\sqrt{2}}$ and $\sqrt{1 + \alpha^2}$. If $\alpha > 1$ then the smallest is $\frac{1-\alpha}{\sqrt{2}}$ the biggest is $\sqrt{1 + \alpha^2}$, note that $|\alpha - 1| \leqslant 2\sqrt{1 + \alpha^2}$.

If $\alpha = 1$ then the smallest value of f is 0, the biggest is $\sqrt{2}$. If $-1 < \alpha < 1$ the the smallest value of f is $\frac{\alpha-1}{\sqrt{2}}$ and the biggest is $\sqrt{1 + \alpha^2}$. If $\alpha = -1$ then the inequalities $-\sqrt{2} = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \leqslant f(x, y) \leqslant f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \sqrt{2}$ hold.

Let $\alpha < -1$. In this case eventually possible extreme values are $\frac{1-\alpha}{\sqrt{2}}$, $\frac{\alpha-1}{\sqrt{2}}$ and $-\sqrt{1 + \alpha^2}$. The biggest is $\frac{1-\alpha}{\sqrt{2}}$, the smallest is $-\sqrt{1 + \alpha^2}$.