Integration in several variables

We remind basic theorems.

Theorem 10.1 (Fubini)

If f is Riemann integrable on the rectangle $[a,b]\times[c,d]$ then

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \int_c^d\left(\int_a^b f(x,y)dx\right)dy = \int_a^b\left(\int_c^d f(x,y)dy\right)dx.$$

Theorem 10.2 (change of the variables formula)

If $\Phi\colon G'\longrightarrow G$ is a diffeomorhpism of an open set G' onto an open set G ($\Phi(u,v)=(x,y)\in G'$), $f\colon G\to\mathbb{R}$ is a Riemann integrable function on a set $A=\Phi(A')$ then

$$\iint_A f(x,y)dxdy = \iint_{A'} f(\Phi(u,v))|\det D\Phi(u,v)|dudv.$$

Both theorems hold in more dimensional case. In the one dimensional case in the second theorem there was no absolute value. This is in fact the same theorem. If φ maps an interval $[a,b]$ onto the interval $[c,d]$ and $\varphi'<0$ then the function φ decreases and we have $\varphi(a)=d$ and $\varphi(b)=c$ and instead of writing $\int_a^b f(\varphi(t))\varphi'(t)dt=\int_{\varphi(a)}^{\varphi(b)} f(u)du=\int_d^c f(u)du$ (as in the previous semester) we can write $\int_a^b f(\varphi(t))|\varphi'(t)|dt=\int_{\varphi(b)}^{\varphi(a)} f(u)du=\int_c^d f(u)du$ – this time we put smaller end of the interval down and the bigger up. This is due to the problems with several variables which we shall not discuss here.

Example 10.3 Let $\Phi(u,v)=(au+bv,cu+dv)$. We have $\Phi(0,0)=(0,0)$, $\Phi(1,0)=(a,c)$, $\Phi(0,1)=(b,d)$ and $\Phi(1,1)=(a+b,c+d)$. The map Φ maps the unit square Q with the vertices $(0,0),(1,0),(1,1),(0,1)$ onto the quadrilateral P with the vertices $(0,0),(a,c),(a+b,c+d),(b,d)$. If it is assumed that $\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc\neq 0$ then the map is one–to–one (injective) and the points $(0,0),(a,c),(a+b,c+d),(b,d)$ do not lie on one straight line: equation of the line L through $(0,0)$ and (a,c) is $cx-ay=0$ so the point (b,d) does not lie on L. Neither $(a+b,c+d)$ does since $c(a+b)-a(c+d)=bc-ad\neq 0$. If we integrate the function 1 (a constant function) over P we obtain the area of P. On the other hand

$$\iint_P dxdy = \iint_Q \begin{vmatrix} a & b\\ c & d\end{vmatrix} dudv = \begin{vmatrix} a & b\\ c & d\end{vmatrix} = |ad-bc|.$$

This example shows that the formula for the change of the variables contains a formula for the area of a parallelogram. □

If we integrate a function which is 1 at each point of the set $A\subseteq R$ and 0 outside A then we obtain the length of A. A may be an interval or a union of finitely many disjoint intervals or enem a union of infinitely many intervals (in this case some additional assumption should be made). If $A\subset\mathbb{R}^2$ is a compact set and f is a function with the value 1 at each point of A and which is 0 outside A then the integral of f over A is the area of A. Analogous statement is true in the three–dimensional space. If we apply the Fubini theorem the results is: the area equals to the integral of lengths of horizontal sections over the appropriate domain. This is not very precise statement. Let us look at examples.

Example 10.4 Area of a triangle. Suppose that the base of a triangle lies on the x–axis and has length $a>0$. Let us assume also that the altitude of the triangle is $h>0$, let the vertex of the triangle outside the horizontal axis be (c, h). If we cut the triangle with straight line consisting of points with the second coordinate $y \in (0, h)$ we obtain an interval of the length $\ell(y) = \frac{h-y}{h} \cdot a$ (the ratio of the lengths of corresponding elements in similar triangles is equal to the scale. We can write $\int_0^h \ell(y)dy = \int_0^h \frac{a(h-y)}{h}dy = -\frac{a}{2h}(h-y)^2\Big|_0^h = -\frac{a}{2h}(0^2 - h^2) = \frac{ah}{2}$. □

Example 10.5 Area of an ellipse. Let $a, b > 0$ and let $E = \{(x, y)\colon \ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\}$. E is an ellipse (if $a = b$ it is a disc of radius a). We shall find the area of E. Let us fix $y \in (-b, b)$ for a moment. Then x satisfies the inequality

$$-\frac{a}{b}\sqrt{b^2 - y^2} \leqslant x \leqslant \frac{a}{b}\sqrt{b^2 - y^2}.$$

The numbers x corresponding to this y form an interval of the length $2\frac{a}{b}\sqrt{b^2 - y^2}$. The area of E is therefore

$$\int_{-b}^{b} 2\frac{a}{b}\sqrt{b^2 - y^2}dy \ \overset{y=b\sin t}{\underset{dy=b\cos t dt}{=\!=\!=}} \ \int_{-\pi/2}^{\pi/2} 2\frac{a}{b} \cdot b^2 \cdot \cos^2 t dt = ab\int_{-\pi/2}^{\pi/2}(\cos(2t) + 1)dy =$$

$$= ab(\tfrac{1}{2}\sin(2t) + t)\Big|_{-\pi/2}^{\pi/2} = ab(2\sin\pi + \pi) = \pi ab.$$

Notice that for $a = b$ we have obtained the formula for the area of a dis of radius a. □

In the same way we obtain formulas for the volumes. The difference is that instead of the length of cross–sections we have to integrate now the area of them. Let us look at some examples.

Example 10.6 Volume of a cone. Let us assume that there is a compact set B contained in the plane $z = 0$ (the plane contains x–axis and y–axis). Let us assume that there is a point $v = (a, b, h)$ with $h > 0$. The cone with the base B and a apex v consists of all straight line segments with one end in B and another one v. If B is a polygon then the cone is a pyramid If B is a disc and v is right above the center of the disc the cone is a real cone (which one can obtain by roteting a right triangle about one of its catheti). So our cone is more general than a traditional cone. We are going to prove that the volume of the cone is $\frac{1}{3}A(B)h$ where $A(B)$ is the area of B. This means that the volume of our cone is one third of the product of the area of its base and of the altitude.

The cross-section on the level $z \in (0, h)$ is a set similar to B the scale of this similarity is $\frac{h-z}{h}$ (the lines through v define the similarity). The area of the cross–section is therefore $A(z) = (\frac{h-z}{h})^2 A(B) = (\frac{h-z}{h})^2 A(0)$. The volume is therefore

$$\int_0^h A(z)dz = \int_0^h \left(\frac{h-z}{h}\right)^2 A(0)dz = -\frac{A(0)(h-z)^3}{3h^2}\Big|_0^h = \frac{1}{3}A(0)h = \frac{1}{3}A(B)h.$$

We are done. □

Example 10.7 Volume of an ellipsoid. Let $a, b, c > 0$ and let $E = \{(x, y, z)\colon \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\}$. Let us fix $z \in (-c, c)$ for a moment. The set defined by the inequality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1 - \frac{z^2}{c^2}\}$$

is an ellipse with area $A(z) = \pi ab\left(1 - \frac{z^2}{c^2}\right)$. This implies that the volume of the ellipsoid is

$$\int_{-c}^{c} A(z)dz = \int_{-c}^{c} \pi ab\left(1 - \frac{z^2}{c^2}\right)dz = \pi ab\left(z - \frac{z^3}{3c^2}\right)\Big|_{-c}^{c} = \frac{4}{3}\pi abc.$$

For $a = b = c$ the ellipsoid becomes a ball of the radius a. □

Example 10.8 An integral. $\int_0^1\int_y^1 \frac{1}{1+x^4}dxdy$ (Temat 28. part 2m) One can use the partial fractions. At first such person factors out the denominator

$$1+x^4=(1+x^2)^2-2x^2=(1-x\sqrt{2}+x^2)(1+x\sqrt{2}+x^2).$$

Then writes $\frac{1}{1+x^4}=\frac{Ax+B}{1-x\sqrt{2}+x^2}+\frac{Cx+D}{1+x\sqrt{2}+x^2}$. This equation is satisfied for all $x\in\mathbb{R}$. We may substitute any number for x. We are looking for the coefficients A,B,C,D. We need four equations. Lets us use $x\in\{0,-1,1\}$ and $\lim\limits_{x\to\infty}\frac{x}{1+x^4}=0$. We get
$1=B+D$,
$\frac12=\frac{-A+B}{1+\sqrt{2}+1}+\frac{-C+D}{1-\sqrt{2}+1}=\frac{-A+B}{2+\sqrt{2}}+\frac{-C+D}{2-\sqrt{2}}=\frac{-A(2-\sqrt{2})+B(2-\sqrt{2})-C(2+\sqrt{2})+D(2+\sqrt{2})}{2}$,
$\frac12=\frac{A+B}{1-\sqrt{2}+1}+\frac{C+D}{1+\sqrt{2}+1}=\frac{A+B}{2-\sqrt{2}}+\frac{C+D}{2+\sqrt{2}}=\frac{A(2+\sqrt{2})+B(2+\sqrt{2})+C(2-\sqrt{2})+D(2-\sqrt{2})}{2}$,
$0=\lim\limits_{x\to\infty}\frac{x}{1+x^4}=\lim\limits_{x\to\infty}\frac{x(Ax+B)}{1-x\sqrt{2}+x^2}+\lim\limits_{x\to\infty}\frac{x(Cx+D)}{1+x\sqrt{2}+x^2}=A+C.$
It implies that $-1=(A-C)\sqrt{2}+(D-B)\sqrt{2}$ and $-1=(A-C)\sqrt{2}-(D-B)\sqrt{2}$. Adding the two equations results in $-2=2\sqrt{2}(A-C)$ so $A-C=-\frac{1}{\sqrt{2}}$ so $A=-\frac{1}{2\sqrt{2}}$ and $C=\frac{1}{2\sqrt{2}}$. Also $D-B=0$ so $B=D=\frac12$. We are ready to write

$$\frac{1}{1+x^4}=\frac{-\frac{1}{2\sqrt{2}}x+\frac12}{1-x\sqrt{2}+x^2}+\frac{\frac{1}{2\sqrt{2}}x+\frac12}{1+x\sqrt{2}+x^2}=\frac{1}{2\sqrt{2}}\left(\frac{-x+\sqrt{2}}{1-x\sqrt{2}+x^2}+\frac{x+\sqrt{2}}{1+x\sqrt{2}+x^2}\right).$$

Now we are ready to integrate. Twice. We obtain (after some standard work)
$\frac{1}{2\sqrt{2}}\int\frac{x+\sqrt{2}}{1+x\sqrt{2}+x^2}dx=\frac{1}{4\sqrt{2}}\ln(1+x\sqrt{2}+x^2)+\frac{1}{2\sqrt{2}}\arctan(1+x\sqrt{2})+\text{const}$ and
$\frac{1}{2\sqrt{2}}\int\frac{-x+\sqrt{2}}{1-x\sqrt{2}+x^2}dx=-\frac{1}{4\sqrt{2}}\ln(1-x\sqrt{2}+x^2)-\frac{1}{2\sqrt{2}}\arctan(1-x\sqrt{2})+\text{const}.$
We can now write
$\int_y^1\frac{1}{1+x^4}dx=\frac{1}{4\sqrt{2}}(\ln(2+\sqrt{2})-\ln(2-\sqrt{2}))+\frac{1}{2\sqrt{2}}(\arctan(1+\sqrt{2})-\arctan(1-\sqrt{2}))-$
$-\frac{1}{4\sqrt{2}}\ln(1+y\sqrt{2}+y^2)-\frac{1}{2\sqrt{2}}\arctan(1+y\sqrt{2})+\frac{1}{4\sqrt{2}}\ln(1-y\sqrt{2}+y^2)+\frac{1}{2\sqrt{2}}\arctan(1-y\sqrt{2}).$
Now it is time for integration. At first we simplify the formula a little bit.. Notice that
$\frac{1}{4\sqrt{2}}(\ln(2+\sqrt{2})-\ln(2-\sqrt{2}))=\frac{1}{4\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}=\frac{1}{4\sqrt{2}}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{1}{4\sqrt{2}}\ln(\sqrt{2}+1)^2=\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1).$
Also $(\sqrt{2}-1)(\sqrt{2}+1)=1$ so if $\tan\alpha=\sqrt{2}-1$ then $\sqrt{2}+1=\cot\alpha=\tan(\frac{\pi}{2}-\alpha)$. This proves that $\arctan(1+\sqrt{2})-\arctan(1-\sqrt{2})=\arctan(1+\sqrt{2})+\arctan(\sqrt{2}-1)=\frac{\pi}{2}-\alpha+\alpha=\frac{\pi}{2}$
Now we can write
(1) $\int_y^1\frac{1}{1+x^4}dx=\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1)+\frac{\pi}{4\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln(1+y\sqrt{2}+y^2)+\frac{1}{4\sqrt{2}}\ln(1-y\sqrt{2}+y^2)-$
$-\frac{1}{2\sqrt{2}}\arctan(1+y\sqrt{2})+\frac{1}{2\sqrt{2}}\arctan(1-y\sqrt{2}).$
Let us start with $\int\ln(1+y\sqrt{2}+y^2)dy=\int\big(\ln(2+2y\sqrt{2}+2y^2)-\ln2\big)\,dy=$
$=\int\big(\ln(1+(y\sqrt{2}+1)^2)-\ln2\big)\,dy \underset{du=\sqrt{2}dy}{\overset{u=y\sqrt{2}+1}{=\!=\!=}} \frac{1}{\sqrt{2}}\int(\ln(1+u^2)-\ln2)du=$
$=-\frac{u\ln2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(u\ln(1+u^2)-\int\frac{2u^2}{u^2+1}du\right)=\frac{u\ln(1+u^2)-u\ln2}{\sqrt{2}}-\int\sqrt{2}du+\int\frac{\sqrt{2}}{u^2+1}du$
$=\frac{u\ln(1+u^2)-u\ln2}{\sqrt{2}}-\sqrt{2}u+\sqrt{2}\arctan u+\text{const}=\frac{u}{\sqrt{2}}\ln\frac{1+u^2}{2}-\sqrt{2}u+\sqrt{2}\arctan u+\text{const}=$
$=(y+\frac{1}{\sqrt{2}})\ln(1+y\sqrt{2}+y^2)-\sqrt{2}(y\sqrt{2}+1)+\sqrt{2}\arctan(y\sqrt{2}+1)+\text{const}=$
$=\frac{y\sqrt{2}+1}{\sqrt{2}}\ln(1+y\sqrt{2}+y^2)-2y+\sqrt{2}\arctan(y\sqrt{2}+1)+\text{Const}.$ The constant has changed.
From the obtained result it follows easily that
$\int_0^1\ln(1+y\sqrt{2}+y^2)dy=\frac{(\sqrt{2}+1)(\ln(2+\sqrt{2})}{\sqrt{2}}-2+\sqrt{2}\big(\arctan(\sqrt{2}+1)-\arctan1\big)=$
$=\frac{(\sqrt{2}+1)(\ln(2+\sqrt{2})}{\sqrt{2}}-2+\frac{\pi\sqrt{2}}{8}.$
We used the formula $\tan\frac{3\pi}{8}=\sqrt{2}+1$ which one can derive using the formula $\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$, namely $\frac{2(\sqrt{2}+1)}{1-(\sqrt{2}+1)^2}=\frac{2(\sqrt{2}+1)}{-2-2\sqrt{2}}=-1=\tan\frac{3\pi}{4}$. This proves that if $0<\alpha<\frac{\pi}{2}$ and $\tan\alpha=\sqrt{2}+1$ then $\alpha=\frac{3\pi}{4\cdot2}=\frac{3\pi}{8}$. Now let us compute

$\int_0^1 \ln(1-y\sqrt{2}+y^2)dy \ \underset{du=-dy}{\overset{u=-y}{=\!=\!=\!=\!=}} \ = \int_{-1}^0 \ln(1+u\sqrt{2}+u^2)du =$

$= \sqrt{2}\arctan 1 - (\frac{(1-\sqrt{2})\ln(2-\sqrt{2})}{\sqrt{2}} + 2 + \sqrt{2}\arctan(1-\sqrt{2}) =$

$= \sqrt{2}\arctan 1 + \frac{(\sqrt{2}-1)\ln(2-\sqrt{2})}{\sqrt{2}} - 2 + \sqrt{2}\arctan(\sqrt{2}-1) = \frac{(\sqrt{2}-1)\ln(2-\sqrt{2})}{\sqrt{2}} - 2 + \frac{3\pi\sqrt{2}}{8}.$

It is time for $\int \arctan(1+y\sqrt{2})dy = y\arctan(1+y\sqrt{2}) - \int \frac{y\sqrt{2}}{1+(1+y\sqrt{2})^2}dy =$

$= y\arctan(1+y\sqrt{2}) - \frac{1}{2}\int \frac{2(y\sqrt{2}+1)}{1+(1+y\sqrt{2})^2}dy + \int \frac{1}{1+(1+y\sqrt{2})^2}dy =$

$= y\arctan(1+y\sqrt{2}) - \frac{1}{2\sqrt{2}}\ln(1+(1+y\sqrt{2})^2) + \frac{1}{\sqrt{2}}\arctan(1+y\sqrt{2}) + \mathrm{const} =$

$= \left(y+\frac{1}{\sqrt{2}}\right)\arctan(1+y\sqrt{2}) - \frac{1}{2\sqrt{2}}\ln(2+2y\sqrt{2}+2y^2) + \mathrm{const} =$

$= \left(y+\frac{1}{\sqrt{2}}\right)\arctan(1+y\sqrt{2}) - \frac{1}{2\sqrt{2}}\ln(1+y\sqrt{2}+y^2) - \frac{1}{2\sqrt{2}}\ln 2 + \mathrm{const} =$

$= \left(y+\frac{1}{\sqrt{2}}\right)\arctan(1+y\sqrt{2}) - \frac{1}{2\sqrt{2}}\ln(1+y\sqrt{2}+y^2) + \mathrm{Const}.$

Now we compute $\int \arctan(1-y\sqrt{2})dy \ \underset{du=-dy}{\overset{u=-y}{=\!=\!=\!=\!=}} \ -\int \arctan(1+u\sqrt{2})du =$

$= -\left(u+\frac{1}{\sqrt{2}}\right)\arctan(1+u\sqrt{2}) + \frac{1}{2\sqrt{2}}\ln(1+u\sqrt{2}+u^2) + \mathrm{const} =$

$= \left(y-\frac{1}{\sqrt{2}}\right)\arctan(1-y\sqrt{2}) + \frac{1}{2\sqrt{2}}\ln(1-y\sqrt{2}+y^2) + \mathrm{const}.$

Now we can write

$\int_0^1 \arctan(1+y\sqrt{2})dy = \left(1+\frac{1}{\sqrt{2}}\right)\arctan(1+\sqrt{2}) - \frac{1}{2\sqrt{2}}\ln(2+\sqrt{2}) - \frac{1}{\sqrt{2}}\arctan 1 =$

$= \left(1+\frac{1}{\sqrt{2}}\right)\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\frac{\pi}{4} - \frac{1}{2\sqrt{2}}\ln(2+\sqrt{2}) = \frac{\pi}{8}\left(3+\frac{1}{\sqrt{2}}\right) - \frac{\ln(2+\sqrt{2})}{2\sqrt{2}}$ and

$\int_0^1 \arctan(1-y\sqrt{2})dy = \left(1-\frac{1}{\sqrt{2}}\right)\arctan(1-\sqrt{2}) + \frac{1}{2\sqrt{2}}\ln(2-\sqrt{2}) + \frac{1}{\sqrt{2}}\arctan 1 =$

$= -\left(1-\frac{1}{\sqrt{2}}\right)\frac{\pi}{8} + \frac{1}{\sqrt{2}}\frac{\pi}{4} + \frac{1}{2\sqrt{2}}\ln(2-\sqrt{2}) = \frac{\pi}{8}\left(\frac{3}{\sqrt{2}}-1\right) + \frac{\ln(2-\sqrt{2})}{2\sqrt{2}}$

Let us return to the formula (1) in order to compute at last the integral

$$\int_0^1\left(\int_y^1 \frac{1}{1+x^4}dx\right)dy = \int_0^1\left(\frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln(1+y\sqrt{2}+y^2) + \frac{1}{4\sqrt{2}}\ln(1-y\sqrt{2}+y^2) -\right.$$
$$\left. - \frac{1}{2\sqrt{2}}\arctan(1+y\sqrt{2}) + \frac{1}{2\sqrt{2}}\arctan(1-y\sqrt{2})\right)dy =$$
$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\int_0^1\Big(\ln(1-y\sqrt{2}+y^2) - \ln(1+y\sqrt{2}+y^2)\Big)\mathrm{d}y +$$
$$+ \frac{1}{2\sqrt{2}}\int_0^1\Big(\arctan(1-y\sqrt{2}) - \arctan(1+y\sqrt{2})\Big)dy =$$
$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\left(\frac{(\sqrt{2}-1)(\ln(2-\sqrt{2})}{\sqrt{2}} - 2 + \frac{3\pi\sqrt{2}}{8} - \frac{(\sqrt{2}+1)(\ln(2+\sqrt{2})}{\sqrt{2}} + 2 - \frac{\pi\sqrt{2}}{8}\right) +$$
$$+ \frac{1}{2\sqrt{2}}\left(\frac{\pi}{8}\left(\frac{3}{\sqrt{2}}-1\right) + \frac{\ln(2-\sqrt{2})}{2\sqrt{2}} - \frac{\pi}{8}\left(3+\frac{1}{\sqrt{2}}\right) + \frac{\ln(2+\sqrt{2})}{2\sqrt{2}}\right) =$$
$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\left(\frac{(\sqrt{2}-1)(\ln(2-\sqrt{2})}{\sqrt{2}} - \frac{(\sqrt{2}+1)(\ln(2+\sqrt{2})}{\sqrt{2}} + \frac{\pi\sqrt{2}}{4}\right) +$$
$$+ \frac{1}{2\sqrt{2}}\left(\frac{\pi}{8}\left(\sqrt{2}-4\right) + \frac{\ln(2-\sqrt{2})}{2\sqrt{2}} + \frac{\ln(2+\sqrt{2})}{2\sqrt{2}}\right) =$$
$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\left(\ln\frac{2-\sqrt{2}}{2+\sqrt{2}} - \frac{1}{\sqrt{2}}\ln(4-2) + \frac{\pi\sqrt{2}}{4}\right) + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{8}\left(\sqrt{2}-4\right) + \frac{\ln(4-2)}{2\sqrt{2}}\right)$$
$$= \frac{1}{2\sqrt{2}}\ln(\sqrt{2}+1) + \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\pi}{16} + \frac{\pi}{16\sqrt{2}}\left(\sqrt{2}-4\right) = \frac{\pi}{8} + \frac{1}{4\sqrt{2}}\ln\frac{(\sqrt{2}-1)(\sqrt{2}+1)^2}{\sqrt{2}+1} = \frac{\pi}{8}.$$

It is not difficult but it takes time. One may integrate by parts. This leads to the result. I hope that all readers understand at this point that before applying a correct method that gives a solution one should try to find another method (it is not always possible) and solve the problem faster with much less effort. In this case a good idea is to change the order of integration. The Fubini theorem allows this. We have

$$\int_0^1\int_y^1 \frac{1}{1+x^4}dxdy = \int_0^1\int_0^x \frac{1}{1+x^4}dydx = \int_0^1 \frac{x}{1+x^4}dx \ \underset{du=2xdx}{\overset{u=x^2}{=\!=\!=\!=\!=}} \ \frac{1}{2}\int_0^1 \frac{1}{1+u^2}du =$$
$$= \frac{1}{2}(\arctan 1 - \arctan 0) = \frac{\pi}{8}.$$

As you the problem may take just few minutes but an idea is necessary. □