Integrals

Integration is an operation converse to differentiation. We start from the definition.

Definition 0.1 *(of a primitive function or an antiderivative or an indefinite integral)*
Let $G \subset \mathbb{R}$ be the union of pairwise disjoint intervals. If $f\colon G \longrightarrow \mathbb{R}$ is a function defined on the set G then each function $F\colon G \longrightarrow \mathbb{R}$ for which the equality $F'(x) = f(x)$ holds for all $x \in G$ is called an antiderivative (or a primitive or an indefinite integral) of f. We write

$$F(x) = \int f(x)dx \quad \blacksquare$$

Obviously if F is an antiderivative of f then for each $C \in \mathbb{R}$ the function $F + C$ is jest an antiderivative of f, too.

Theorem 0.2 *(uniqueness of an antiderivative)*
If $f\colon P \longrightarrow \mathbb{R}$ P is a function defined on an interval P and F_1, F_2 are its antiderivatives then there exists a number $C \in \mathbb{R}$ such that the equality $F_2(x) = F_1(x) + C$ holds for all $x \in P$.

Proof. The derivative of $F_2 - F_1$ is equal to 0 for all $x \in P$. Therefore the function $F_2 - F_1$ is constant (by the Lagrange mean value theorem). $\blacksquare$

For a function defined on the union of at least two disjoint intervals the theorem is not true. The function $\ln|x|$ is an antiderivative of the function $\frac{1}{x}$. Let us denote $F_1(x) = \ln|x|$ and $F_2(x) = 1 + \ln x$ dla $x > 0$ and $F_2(x) = \ln(-x)$ for $x < 0$. Clearly $F_2'(x) = \frac{1}{x}$ for all $x \neq 0$ so F_2 is an antiderivative of ln so is F_1. The difference $F_2 - F_1$ assumes two distinct values namely 0 for $x < 0$ and 1 for $x > 0$. This difference is not constant but **it is constant on each interval contained in the domain of f**. Sometimes we shall call such a function **locally constant.** In the sequel we shall write:

$$\int f(x)\ dx = F(x) + C,$$

if F is an antiderivative of f and C will be understood as a locally constant function.

Example 0.3 $\int\ dx = x + C$. $\blacksquare$

Example 0.4 $\int e^x\ dx = e^x + C$. $\blacksquare$

Example 0.5 $\int \cos x\ dx = -\sin x + C$. $\blacksquare$

Example 0.6 $\int \sin x\ dx = \cos x + C$. $\blacksquare$

Example 0.7 $\int \frac{1}{x}\ dx = \ln|x| + C$. $\blacksquare$

Example 0.8 $\int x^a\ dx = \frac{1}{a+1}x^{a+1} + C$ for $a \neq -1$ and all x for which the value x^a is defined (if $a > 0$ is a rational number of the form $\frac{k}{2m+1}$, k, $m \in \mathbb{Z}$ then the domain ot this function is $\mathbb{R}$; if $a < 0$ is a rational number of the form $\frac{k}{2m+1}$, k, $m \in \mathbb{Z}$ then its domain consists of all reals $x \neq 0$; if $a > 0$ is either a rational number of another form or an irrational number then the domain is $[0, \infty)$; if $a < 0$ is a rational number that cannot be written as $\frac{k}{2m+1}$, k, $m \in \mathbb{Z}$ or an irrational number then its domain is $(0, \infty)$). $\blacksquare$

Example 0.9 $\int \frac{1}{1+x^2}\ dx = \operatorname{arctg} x + C$. $\blacksquare$

These formulas should be memorized by all students. This means that everybody should evaluate quite a few integrals.

There are functions without any antiderivative at all. Niech $f(x) = 0$ for $x \leqslant 0$ and $f(x) = 1$ for $x > 0$. Suppose that F is an antiderivative of f. Then for all $x \leqslant 0$ the equality $F'(x) = 0$ holds. Therefore the function F is constant of the half line $(-\infty, 0]$. For $x > 0$ we have $F'(x) = 1$ so there is a number c such that $F(x) = x + c$ for all $x > 0$. Since F is differentiable on $\mathbb{R}$ in particular at the point 0 it is continuous at this point therefore $F(0) = c$. This implies that $F(x) = c$ for all $x \leqslant 0$ and $F(x) = x + c$ for all $x > 0$. Unfortunately this function has no derivative at 0: the left–hand side derivative at 0 equals 0 while the right–hand side derivative equals 1. This is an illustration of the more general statement. One can show that if the derivative of some function defined on an interval exists for all points of the interval then this derivative has the intermediate value property, called also Darboux property.

We shall shown that continuous functions have antiderivatives.

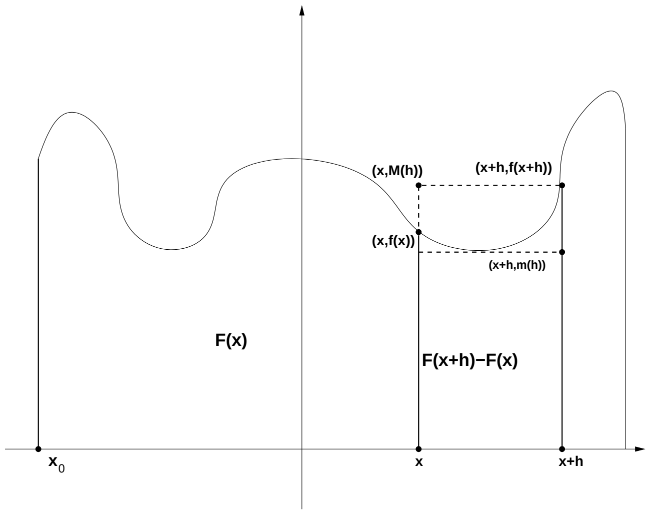
Theorem 0.10 (about an antiderivative of a continuous function on an interval) If $f\colon P \longrightarrow \mathbb{R}$ a continuous function defined on an interval P then there exists a function $F\colon R \longrightarrow \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in P$.

Proof. (sketch). At first we assume that $f(x) > 0$ for all$x$. Let $x_0 \in P$ and $x \geqslant x_0$. Let $F(x)$ be the area of the region bounded from below by the segment $[x_0, x]$, bounde from above by the graph of f, bounded from the left by the vertical line through $(x_0, 0)$ and from the right by the line through $(x, 0)$. If $x < x_0$ then instead of the area we consider a negative number whose absolute value is the corresponding area. We are going to show that $F'(x) = f(x)$ for $x > x_0$ leaving the later case to students. Let $h > 0$ be so small that $x+h \in P$. Let

$$M(h) = \sup\{f(t)\colon \quad x \leqslant t \leqslant x + h\} \quad \text{and} \quad m(h) = \inf\{f(t)\colon \quad x \leqslant t \leqslant x + h\}.$$

The number $F(x + h) - F(x)$ is the area of the region contained in the rectangle of height $M(h)$ with bottom side $[x, x + h]$ and contains the rectangle of height $m(h)$ with bottom side $[x, x + h]$. Therefore

$$m(h) \leqslant \frac{F(x+h) - F(x)}{h} \leqslant M(h).$$



The function f is continuous. Therefore

$$\lim_{h\to 0^+} m(h) = f(x) = \lim_{h\to 0^+} M(h).$$

This implies that $\lim\limits_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x)$. Small changes in the above reasoning prove that $\lim\limits_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = f(x)$. If the function f attains negative values one can add to it a positive number c so big that $f(t)+c>0$ for all t that lie between x and $x+h$. This is possible because a function continuous on a closed interval is bounded. We consider only a part of the domain of f (the interval P does not need to be closed) because it suffices for this proof. ■

The above proof shows a connection of integrals and areas.

Corollary 0.11 (from the proof of the existence of an antiderivative)
If a function f is continuous and non–negative on an interval $[a,b]$ and F is its antiderivative then the area of the region $A=\{(x,y)\colon \quad a\leqslant x\leqslant b,\ 0\leqslant y\leqslant f(x)\}$ i.e. „the area under the graph of f"is equal to

$$F(b)-F(a).$$

Proof. In the proof of the existence of an antiderivative we have indicated an antiderivative F of f for which the above formula holds. By the uniqueness theorem the difference $F(b)-F(a)$ does not depend on the choice of the antiderivative for the antiderivatives differ only by a constant. ■

Definition 0.12 Newton's integral)
The definite integral of $f\colon [a,b]\longrightarrow\mathbb{R}$ is the number $F(b)-F(a)$ where F denotes an antiderivative of f. It is denoted by

$$\int_a^b f(x)dx=F(b)-F(a).\ \blacksquare$$

We shall also write $F(b)-F(a)=F(x)\Big|_a^b$ so $\displaystyle\int_a^b f(x)dx=F(x)\Big|_a^b$.

Example 0.13 $\int_a^b dx=b-a$ since the area of the rectangle with bottom equal to $b-a$ and the altitude 1 equals $b-a$. ■

Example 0.14 $\int_a^b xdx=\frac{b^2-a^2}{2}$ for $\int xdx=\frac12x^2+C$. It is an expected result because if $0\leqslant a$ then $\int_a^b xdx$ is the area o the trapezium with bases a and b and the height $b-a$ so it is $\frac12(b+a)(b-a)$. If $b\leqslant 0$ then the bases of the trapezium are $|a|=-a$ and $|b|=-b$, the height is $b-a$ therefore the integral is a number opposite to the area so it equals $-\frac12\big(-b+(-a)\big)(b-a)=\frac{b^2-a^2}{2}$. There is one more case, namely $a<0<b$. We may see that $\int_a^b xdx=\int_a^0 xdx+\int_0^b xdx$. In this case the integral is the difference of the areas of the two right isosceles triangles with the legs $|a|$ and b. These areas are $\frac12b^2$ and $\frac12a^2$ so the integral equals $\frac12(b^2-a^2)$. ■

Example 0.15 $\int_0^a x^2dx=\frac{a^3}{3}$, *for* $\int x^2dx=\frac13x^3+C$. Therefore „the area under the parabola" equals to $\frac13$ of the area of the rectangle with the vertices $(0,0)$, $(a,0)$, (a^2,a), $(0,a^2)$. The formula was known to Archimedes but his derivation was quite difficult for no integrals were known and only few people were able to understand it. ■

Theorem 0.16 (integral of the sum)
If both functions f and g have the antiderivatives then the functions $f\pm g$ also have the antiderivatives and the following formulas hold:

$$\int\big(f(x)\pm g(x)\big)dx=\int f(x)dx\pm\int g(x)dx$$

and

$$\int_a^b\big(f(x)\pm g(x)\big)dx=\int_a^b f(x)dx\pm\int_a^b g(x)dx.$$

Proof. All formulas follow immediately from the properties of the derivatives. ■

Theorem 0.17 (integral of the product of a number and a function)
If the function f has an antiderivative and c is a real number then the function cf also has an antiderivative and:

$$\int cf(x)dx = c\int f(x)dx.$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

Proof. These formulas follow right away from the properties of the derivatives. ■

Integrating of a product of two functions is usually much harder. There are two important formulas which can make it easier.

Theorem 0.18 (integration by parts)
Let the functions f and g will be continuously differentiable. Then

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

For the definite integral one has

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx = = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

It is an immediate consequence of the product rule for the derivatives. ■

Theorem 0.19 (integration by substitution)
Let the functions f and g' be continuous and let F be an antiderivative of f. Then:

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

For the definite integral one has

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} F(y)dy = F(g(b)) - f(g(a)).$$

Proof. This formula follows immediately from the chain rule ■

We shall show soon how these theorems may be applied.

On the notation. Instead of $g'(x)dx$ we shall frequently write $dg(x)$; if $y = g(x)$ then $dy = g'(x)dx = dg(x)$. ■

Note that g is a one–to–one function so there exists an inverse function g^{-1} then the equalities $y = g(x)$ and $x = g^{-1}(y)$ are equivalent. Then $dx = d(g^{-1})(y) = (g^{-1})'(y)dy$. Inverse function derivative rule tells us that $\big(g^{-1}\big)'(y) = \big(g^{-1}\big)'\big(g(x)\big) = \frac{1}{g'(x)}$. Therefore $dx = d\big(g^{-1}\big)(y) = \frac{1}{g'(x)}dy$. If one looks at the formula $dy = dg(x) = g'(x)dx$, then he/she thinks that the statement is obvious. It is not so. The symbols dx, dy do not denote numbers in fact we never defined them. $\frac{dy}{dx}$ denotes the derivative of y withe respect to x but dy alone was not defined. Therefore if one wants to use some rules she/he must justify the rules. It turned out that if $dy = g'(x)dx$ then $dx = \frac{1}{g'(x)}dy$ also but to justify it we applied the formula dor the derivative of an inverse function. The symbol $\frac{dy}{dx}$ denotes the derivative not a quotient. But it turned out that sometimes it may be treated as a ratio. This is convenient but it should be applied with some caution. If $x = h(t)$ then along with the equality $dy = g'(x)dx$ we have also $dx = h'(t)dt$. We would like to conclude that $dy = g'(x)h'(t)dt$. This is allowed by the Chain Rule: $y = g\big(h(t)\big)$ so $dy = (g \circ h)'(t)dt$

and $\big(g\circ h\big)'(t) = g'\big(h(t)\big)h'(t)$. Therefore $dy = g'\big(h(t)\big)h'(t)dt$. Again it is very similar to ratios:

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt},$$

that is:

$$dy = (g\circ h)'(t)dt = g'(h(t))h'(t)dt = \frac{dy}{dx}\cdot\frac{dx}{dt}\cdot dt.$$

Usually the formula for integration by parts is written as

$$\int f(x)g'(x)dx = \int fdg = fg - \int gdf = f(x)g(x) - \int g(x)f'(x)dx$$

and the formula for integration by substitution:

$$\int f(g(x))g'(x)dx = \int f(y)dy.$$

Example 0.20 $\int e^{2x}dx \;\underset{dy=2dx}{\overset{y=2x}{=\!=\!=}}\; \int e^y\frac{1}{2}dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{2x} + C$. ■

Example 0.21 $\int xe^{x^2}dx \;\underset{dy=2xdx}{\overset{y=x^2}{=\!=\!=}}\; \int e^y\frac{1}{2}dy = \frac{1}{2}\int e^y dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C$. ■

Example 0.22 $\int \operatorname{tg} x dx = \int \frac{\sin x}{\cos x}dx \;\underset{dy=-\sin x dx}{\overset{y=\cos x}{=\!=\!=}}\; -\int \frac{1}{y}dy = -\ln|y| + C = -\ln|\operatorname{tg} x| + C$. ■

Example 0.23 $\int \operatorname{tg} x dx = \int \frac{\sin x}{\cos x}dx = \int \sin x\cdot\frac{1}{\cos x}dx \;\underset{\text{by parts}}{\overset{\text{integration}}{=\!=\!=}}\; (-\cos x)\cdot\frac{1}{\cos x} -$
$-\int(-\cos x)\cdot\frac{\sin x}{\cos^2 x}dx = -1 + \int \operatorname{tg} x$. This integration by parts did not help us at all. Moreover someone could have thought that the result implies $0 = -1$. It is not so because an indefinite integral is defined up to a constant. The obtained formula tells us nothing.

Example 0.24 $\int\sqrt{r^2 - x^2}dx \;\underset{dx=r\cos t dt}{\overset{x=r\sin t}{=\!=\!=}}\; \int\sqrt{r^2 - r^2\sin^2 t}\, r\cos t dt = \int\sqrt{r^2\cos^2 t}\; r\cos t dt =$
$= \int r^2\cos^2 t dt = r^2\int\frac{1+\cos 2t}{2}dt \;\underset{du=2dt}{\overset{u=2t}{=\!=\!=}}\; r^2\int\frac{1+\cos u}{2}\cdot\frac{du}{2} = \frac{r^2}{2}\left(\frac{u}{2} + \frac{\sin u}{2}\right) + C =$
$= \frac{r^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{r^2}{2}\left(t + \sin t\cos t\right) + C = \frac{r^2}{2}\arcsin\frac{x}{r} + \frac{x}{2}\sqrt{r^2 - x^2} + C$ – in the above we have set $x = r\sin t$, this has been legal because if $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ then x assumes all values from the interval $[-r, r]$ and $\cos t \geqslant 0$ so $\sqrt{\cos^2 t} = \cos t$. ■

Example 0.25 The results of the previous example yields
$\int_{-r}^{r}\sqrt{r^2 - x^2}dx = \frac{1}{2}\left[r^2\arcsin\frac{r}{r} + r\sqrt{r^2 - r^2} - r^2\arcsin\frac{-r}{r} - (-r)\sqrt{r^2 - (-r)^2}\right] =$
$= \frac{1}{2}\left[2r^2\arcsin 1\right] = r^2\frac{\pi}{2} = \frac{\pi r^2}{2}$.
The integral turned out to the half of the area od a circle of radius r. It is so because the graph of the function $\sqrt{r^2 - x^2}$ is a semicircle centered at $(0, 0))$ of radius r so „the area under the graph” is half of the area of a circle of radius r. ■

Example 0.26 $\int xe^x dx = \int x\,(e^x)'\,dx \;\underset{\text{by parts}}{\overset{\text{integration}}{=\!=\!=}}\; xe^x - \int(x)'e^x dx = xe^x - \int e^x dx = xe^x - e^x + C$. ■

Example 0.27 $\int x^2e^x dx = \int x^2\,(e^x)'\,dx \;\underset{\text{by parts}}{\overset{\text{integration}}{=\!=\!=}}\; x^2e^x - \int (x^2)'\,e^x dx = x^2e^x - \int 2xe^x dx =$
$= x^2e^x - 2\int xe^x dx \;\underset{\text{example}}{\overset{\text{previous}}{=\!=\!=}}\; x^2e^x - 2\left(xe^x - e^x\right) + C = e^x(x^2 - 2x + 2) + C.$

In this example we have used the result of the previous one. In the same way we can evaluate the integrals of x^3e^x, x^4e^x etc. Each integration by parts lowers by 1 the degree of the polynomial which is multiplied by e^x so we end up with the integral $\int e^x dx$. ■

Example 0.28 $\int xe^{3x}dx = \int x\left(\frac{1}{3}e^{3x}\right)' dx \overset{\text{integration}}{\underset{\text{by parts}}{=\!=\!=}} \frac{1}{3}xe^{3x} - \frac{1}{3}\int (x)'e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx =$
$=\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.$

The last integration by substitution ($y = 3x$) was so obvious that was not mentioned, it was simply done. ■

Example 0.29 $\int x\cos(5x)dx = \int x\left(\frac{1}{5}\sin(5x)\right)' dx \overset{\text{integration}}{\underset{\text{byparts}}{=\!=\!=}} \frac{1}{5}x\sin(5x) - \frac{1}{5}\int (x)'\sin(5x)dx =$
$=\frac{1}{5}x\sin(5x) - \frac{1}{5}\int \sin(5x)dx = \frac{1}{5}x\sin(5x) - \frac{1}{5}\left(-\frac{1}{5}\cos(5x)\right) + C = \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C.$ ■

Example 0.30 $\int \ln x dx \overset{\text{integration}}{\underset{\text{by parts}}{=\!=\!=}} x\ln x - \int x(\ln x)^{\cdot} = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx =$
$= x\ln x - x + C.$ ■

Example 0.31 $\int \arcsin x dx \overset{\text{integration}}{\underset{\text{by parts}}{=\!=\!=}} x\arcsin x - \int x(\arcsin x)^{\cdot} = x\arcsin x - \int x\frac{1}{\sqrt{1-x^2}}dx =$
$\overset{y=1-x^2}{\underset{dy=-2xdx}{=\!=\!=}} x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{y}}dy = x\arcsin x + \frac{1}{2}\int y^{-1/2}dy = x\arcsin x +$
$+ \frac{1}{2(1+(-1/2))}y^{1+(-1/2)} + C = x\arcsin x + \frac{1}{4}\left(1-x^2\right)^{1/2} + C = x\arcsin x + \frac{1}{4}\sqrt{1-x^2} + C.$ ■

Example 0.32 $\int \cos x\sin x\ dx = \frac{1}{2}\int \sin(2x)\ dz \overset{y=2x}{\underset{dy=2dx}{=\!=\!=}} \frac{1}{4}\int \sin y\ dy = -\frac{1}{4}\cos y + C =$
$= -\frac{1}{4}\cos(2x) + C$. Another way of evaluating the same integral.
$\int \cos x\sin x\ dx \overset{\text{integration}}{\underset{\text{by parts}}{=\!=\!=}} \sin x\cdot\sin x - \int \sin x\cos x\ dx$ so $2\int \cos x\sin x\ dx = \sin x\cdot\sin x + C$ hence $\int \cos x\sin x\ dx = \frac{1}{2}\sin x\cdot\sin x + \frac{C}{2}$ we solved a linear equation with an unknown $\int \cos x\sin x\ dx$. Once again the same integral.
$\int \cos x\sin x\ dx \overset{\text{integration}}{\underset{\text{by parts}}{=\!=\!=}} = \cos x(-\cos x) - \int(-\sin x)(-\cos x)\ dx = -\cos^2 x - \int \sin x\cos x\ dx$ so $2\int \cos x\sin x\ dx = -\cos^2 x + C$ hence $\int \cos x\sin x\ dx = -\frac{1}{2}\cos^2 x + \frac{C}{2}$.
Show that the results in all three cases in fact coincide. To this end you may use the formula $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. ■

Example 0.33 $\int \frac{dx}{2x-3} \overset{y=2x-3}{\underset{dy=2xdx}{=\!=\!=}} \int \frac{1}{y}\cdot\frac{1}{2}dy = \frac{1}{2}\ln|y| + C = \frac{1}{2}\ln|2x-3| + C.$ ■

Example 0.34 $\int \frac{x}{x^2+3x+2}dx = \int \frac{x}{(x+1)(x+2)}dx.$

One may expect that the expression $\frac{x}{x^2+3x+2}$ may be written as a sum of fractions of the form $\frac{A}{x+1}$ or $\frac{B}{x+2}$. The equation $\frac{x}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$ is satisfied iff $x = A(x+2) + B(x+1)$ for all real numbers except for -1 and -2.[1] Therefore $A + B = 1$ and $2A + B = 0$ so $A = -1$ and $B = 2$. So:

$$\int \frac{x}{x^2+3x+2}dx = \int \frac{-1}{x+1}dx + \int \frac{2}{x+2}dx = -\ln|x+1| + 2\ln|x+2| + C = \ln\frac{(x+2)^2}{|x+1|} + C.$$ ■

[1] In fact the equality must hold also for 1 and 2 because the functions x and $A(x+2)+B(x+1)$ are continuous at all points , $x = -1$ and $x = -2$ among them.