Exam questions

Let us start with reminding a theorem:

If $(A,B,C) \neq (0,0,0)$ then the equation $Ax+By+Cz+D=0$ equivalent to $Ax+By+Cz=-D$ describes a plane perpendicular to the vector (A,B,C).

This a particular case of the following theorem:

Let g be a C^1 function defined on an open subset of $\mathbb{R}^k$, $c \in \mathbb{R}$, $M = \{\boldsymbol{x} \in \mathbb{R}^k : \quad g(\boldsymbol{x}) = c\}$, $\mathbf{v}$ a vector tangent to M at point $\mathbf{p} \in M$. Then $\nabla g(\mathbf{p}) \cdot \mathbf{v} = 0$. If $\nabla g(\mathbf{p}) \neq \mathbf{0}$ then the converse is true: if $\mathbf{v} \cdot \nabla g(\mathbf{p})$ then the vector $\mathbf{v}$ is tangent to M at $\mathbf{p}$ (i.e. $\mathbf{v} \in T_{\mathbf{p}}M$).

3. Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be given by the formula $g(x,y,z) = 2y\ln(z-2) - xyz + 3x$.

(a) Show that there is a neighborhood of the point $(x,y) = (1,1)$ on which one can define a function $z = z(x,y)$ of class C^1 such that $z(1,1) = 3$ and $g(x,y,z(x,y)) = 0$. Calculate $\frac{\partial z}{\partial x}(1,1)$ and $\frac{\partial z}{\partial y}(1,1)$.

(b) Find the equation of the tangent plane to the surface
$$M = \{(x,y,z) \in \mathbb{R}^3 : \quad g(x,y,z) = 0\}$$
at the point $(1,1,3)$.

Let me start with an approximate formula
$$g(x,y,z) \approx g(1,1,3) + \tfrac{\partial g}{\partial x}(1,1,3)(x-1) + \tfrac{\partial g}{\partial y}(1,1,3)(y-1) + \tfrac{\partial g}{\partial z}(1,1,3)(z-3)$$
therefore $\quad 2y\ln(z-2) - xyz + 3x \approx 0 + 0\cdot(x-1) + (-3)\cdot(y-1) + 1\cdot(z-3)$
because $\frac{\partial g}{\partial x}(x,y,z) = -yz+3$, $\frac{\partial g}{\partial y}(x,y,z) = 2\ln(z-2) - xz$ and $\frac{\partial g}{\partial z}(x,y,z) = \frac{2y}{z-2} - xy$
so $\frac{\partial g}{\partial x}(1,1,3) = 0, \quad \frac{\partial g}{\partial y}(1,1,3) = -3 \quad$ and $\quad \frac{\partial g}{\partial z}(1,1,3) = 1.$
One may write $\nabla g(1,1,3) = (0,-3,1)$.
We want to solve the equation $g(x,y,z) = 0$ for z. This may look hopeless but approximate equation $0 = 0 + 0\cdot(x-1) + (-3)\cdot(y-1) + 1\cdot(z-3) = -3(y-1) + (z-3)$ is easy: $z = 3y$.
The implicit function theorem tells us that this action is legal. More precisely: since $\frac{\partial g}{\partial z}(1,1,3) = 1 \neq 0$ there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that if $|x-1| < \delta$ and $|y-1| < \delta$ then there is exactly one z such that $|z-3| < \varepsilon$ and $g(x,y,z) = 0$. In other words there is a function which assigns to each pair of numbers x,y with $|x-1| < \delta$ and $|y-1| < \delta$ a number $z \in (3-\varepsilon, 3+\varepsilon)$ so there is a map from $(1-\delta,1+\delta)\times(1-\delta,1+\delta)$ into $(3-\varepsilon,3+\varepsilon)$. The map $(x,y) \mapsto z$ is of the same class of differentiability as g, in the case under consideration it is a C^∞ map.
Let me say at this moment that the beginning (from „Let me …” until „The implicit function … is unnecessary. I included it only to tell you where from the implicit function theorem comes from. It is not a question how to prove, it is a question of what should be proved.
Now we know that z is a differentiable function of (x,y). Therefore we are allowed to differentiate the equation $0 = g(x,y,z(x,y))$ relative to x and to y. We must use the chain rule of course. We obtain
$$0 = \tfrac{\partial}{\partial x}\Big(g(x,y,z(x,y))\Big) = \tfrac{\partial g}{\partial x}(x,y,z(x,y)) + \tfrac{\partial g}{\partial z}(x,y,z(x,y)) \cdot \tfrac{\partial z}{\partial x}(x,y).$$

Now we substitute 1 for x and for y and 3 for $z(1,1)$:
$0=\frac{\partial g}{\partial x}(1,1,3)+\frac{\partial g}{\partial z}(1,1,3)\cdot\frac{\partial z}{\partial x}(1,1)=0+1\cdot\frac{\partial z}{\partial x}(1,1)$. The last step is to solve this equation for $\frac{\partial z}{\partial x}(1,1)$. The result is $\frac{\partial z}{\partial x}(1,1)=0$.
Now we repeat the calculations for $\frac{\partial}{\partial y}$. It goes like this

$$0=\frac{\partial}{\partial y}\Big(g(x,y,z(x,y))\Big)=\frac{\partial g}{\partial y}(x,y,z(x,y))+\frac{\partial g}{\partial z}(x,y,z(x,y))\cdot\frac{\partial z}{\partial y}(x,y).$$

Now we substitute 1 for x and for y and 3 for $z(1,1)$:
$0=\frac{\partial g}{\partial y}(1,1,3)+\frac{\partial g}{\partial z}(1,1,3)\cdot\frac{\partial z}{\partial y}(1,1)=-3+1\cdot\frac{\partial z}{\partial y}(1,1)$. The last step is to solve this equation for $\frac{\partial z}{\partial y}(1,1)$. The result is $\frac{\partial z}{\partial y}(1,1)=3$.
Now part b. The tangent plane contains the point $(1,1,3)$ and it is perpendicular to the $\nabla g(1,1,3)=(0,-3,1)$ Therefore its equation is $0\cdot x+(-3)\cdot y+1\cdot z=0\cdot 1+(-3)\cdot 1+1\cdot 3=0$, in short $-3y+z=0$. The end of this story. $\square$

4. Let $\quad M=\{(x,y,z)\in\mathbb{R}^3:\quad (x-1)^2+y^2+z^2=1\quad\text{and}\quad x^2-y^2=1\}$.
Show that M is a one dimensional manifold and that the maximum and minimum values of $f_{|M}$ where $f(x,y,z)=x+y$ occur when $z=0$.
Let $g_1(x,y,z)=(x-1)^2+y^2+z^2-1$ and $g_2(x,y,z)=x^2-y^2-1$. The following formulas hold:

$$\nabla g_1(x,y,z)=2(x-1,y,z)\quad\text{and}\quad\nabla g_2(x,y,z)=2(x,-y,0).\tag{1}$$

Now we shall prove that these gradients are linearly independent for $(x,y,z)\in M$.
If $(0,0,0)=c_1\cdot 2(x-1,y,z)+c_2\cdot 2(x,-y,0)=2(c_1(x-1)+c_2x,c_1y-c_2y,c_1z)$ then either $z=0$ or $c_1=0$.

If $z=0$ then $(x-1)^2+y^2=1$ and $x^2-y^2=1$ so $(x-1)^2+x^2=2$ that is $2x^2-2x-1=0$. This implies that $x=\frac12(1\pm\sqrt3)$ and $x^2=1+y^2\geqslant 1$ so

$$x=\frac{1}{2}(1+\sqrt{3}).$$

Therefore $1=x^2-y^2=\left(\frac12(1+\sqrt3)\right)^2-y^2=\frac14(4+2\sqrt3)-y^2=1+\frac12\sqrt3-y^2$. This implies that $y^2=\frac{\sqrt3}{2}$. There are two such points

$$\left(\frac{1}{2}(1+\sqrt{3}),\sqrt[4]{\frac{3}{4}},0\right)\quad\text{and}\quad\left(\frac{1}{2}(1+\sqrt{3}),-\sqrt[4]{\frac{3}{4}},0\right).$$

From the equation $c_1y-c_2y=(c_1-c_2)\left(\pm\sqrt[4]{\frac34}\right)=0$ it follows now that $c_1=c_2$ and in view of this equality we obtain $0=c_1(x-1)+c_2x=c_1(2x-1)=c_1\cdot\sqrt3$ so $c_1=0$. In this case the gradients are linearly independent.

If $c_1=0$ then $(0,0,0)=c_1\cdot 2(x-1,y,z)+c_2\cdot 2(x,-y,0)=2(c_2x,-c_2y,0)$. If $c_2\neq 0$ then $x=0=y$ contrary to $x^2-y^2=1$. Therefore $c_2=0$. This proves that the gradients are linearly independent also in this case. Therefore M is a manifold.

We can apply the Lagrange theorem. This means that if the function $f_{|M}$ attains its maximal or minimal value at some point (x,y,z) then there are numbers λ_1,λ_2 such that $(1,1,0)=\nabla f(x,y,z)=\lambda_1\nabla g_1(x,y,z)+\lambda_2\nabla g_2(x,y,z)=2(\lambda_1(x-1)+\lambda_2x,\lambda_1y-\lambda_2y,\lambda_1z)$. Either $z=0$ or $\lambda_1=0$. If $\lambda_1=0$ then $(1,1,0)=2(\lambda_2x,-\lambda_2y,0)$ so $x=-y$ i.e. $x+y=0$ contrary to $1=x^2-y^2=(x+y)(x-y)$. If $z=0$ then we are done. $\square$
We were not asked to find max $f_{|M}$ nor min $f_{|M}$ so we are done. We can find them anyway. We already know that if an extreme value is attained then $z=0$ so $(x-1)^2+y^2=1$

and $x^2 - y^2 = 1$. This as we know implies that either $(x,y,z) = \left(\frac{1}{2}(1+\sqrt{3}), \sqrt[4]{\frac{3}{4}}, 0\right)$ or $(x,y,z) = \left(\frac{1}{2}(1+\sqrt{3}), -\sqrt[4]{\frac{3}{4}}, 0\right)$. In the first case $x+y = \frac{1}{2}(1+\sqrt{3}) + \sqrt[4]{\frac{3}{4}}$ in the second case $x+y = \frac{1}{2}(1+\sqrt{3}) - \sqrt[4]{\frac{3}{4}}$. The second number is smaller than the first one. Do we know that $\max f_{|M}$ or $\min f_{|M}$ is attained? The set M is bounded because it is a subset of a sphere of radius 1 centered at $(1,0,0)$. It is closed, so it is compact. Therefore the strict bounds are values of $f_{|M}$. This means that $\max f_{|M} = \frac{1}{2}(1+\sqrt{3}) + \sqrt[4]{\frac{3}{4}}$ and $\min f_{|M} = \frac{1}{2}(1+\sqrt{3}) - \sqrt[4]{\frac{3}{4}}$.

At the end I want to say something about the set M just to mention that we do not need to understand geometry of M in order to solve the problem. The manifold M is not contained in any plane because the points $\left(\frac{1}{2}(1+\sqrt{3}), \sqrt[4]{\frac{3}{4}}, 0\right)$, $\left(\frac{1}{2}(1+\sqrt{3}), -\sqrt[4]{\frac{3}{4}}, 0\right)$, $(1,0,1)$ and $(1,0,-1)$ lie on M and it is easy to see that there is no plane containing the four points.

Three problems.

1. Find the smallest and the biggest value of f defined as $f(x,y,z) = xyz$ on the set $M = \{(x,y,z)\colon \quad x^2+y^2+z^2 \leqslant 9 \quad \text{and} \quad x+y+z = 5\}$.
2. Prove that
$$x_1y_1+x_2y_2+x_3y_3+\dots+x_7y_7 \leqslant (x_1^3+x_2^3+x_3^3+\dots+x_7^3)^{1/3}(y_1^{3/2}+y_2^{3/2}+y_3^{3/2}+\dots+y_7^{3/2})^{2/3}$$
provided that $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, \dots, x_7 \geqslant 0,\ y_1 \geqslant 0, y_2 \geqslant 0, y_3 \geqslant 0, \dots, y_7 \geqslant 0$.
Hint. What happens if $x_1, x_2, x_3, \dots, x_7$ are replaced with $tx_1, tx_2, tx_3, \dots, tx_7$ where t is a positive number? What about y's?
3. Find the greatest and the least value of f where $f(x,y,z) = \frac{x+y}{3x^2+y^2+12}$ on the set $A = \{(x,y)\colon \quad x \geqslant 0\}$.

Solutions.

1. The equation $x+y+z=5$ describes the plane perpendicular to the vector $(1,1,1)$. On this plane the function f is neither bounded above nor below because
$(-n)+(-n)+(2n+5) = 5$ and $\lim\limits_{n\to\infty} (-n)(-n)(2n+5) \xrightarrow[n\to\infty]{} \infty$ and
$n+n+(-2n+5) = 5$ and $\lim\limits_{n\to\infty} n\cdot n\cdot(-2n+5) \xrightarrow[n\to\infty]{} -\infty$.
The second condition changes the situation a lot. The set M is compact because it is closed and bounded (it is contained in the ball of radius 5 centered at $(0,0,0)$). The function f is continuous so its $\sup_M F$ and $\inf_M f$ are attained. If it is attained in a point (x,y,z) with $x^2+y^2+z^2 < 9$ then by the Lagrange theorem there exists a number λ such that
$$(yz, xz, xy) = \nabla f(x,y,z) = \lambda\nabla(x+y+z-5) = \lambda(1,1,1).$$
This implies that $yz = xz = xy$. At least one of the numbers x,y,z is different from 0 because their sum is 5. Since $5^2+0^2+0^2 > 9$ we may assume that at most one these numbers equals 0. Let $x=0$. From $yz = xy$ and $y \neq 0$ it follows that $z = 0$, a contradiction. We proved that $xyz \neq 0$ (no of the numbers x,y,z is 0). But this implies that they all are equal so $x = y = z = \frac{5}{3}$. $(\frac{5}{3})^2 + (\frac{5}{3})^2 + (\frac{5}{3})^2 = \frac{25}{3} < 9$. We

proved that one of the extreme values of f may be $\frac{125}{3} = f\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$. This is the only possibility for $x^2+y^2+z^2<9$.

Let us assume now that $x+y+z=5$ and $x^2+y^2+z^2=9$. The gradients are $(1,1,1)$ and $2(x,y,z)$. They are linearly dependent iff $x=y=z$ but this not possible for points satisfying both equations. This proves that if an extreme value is attained at such point then there exist numbers λ_1, λ_2 such that the following equations are satisfied

$$\begin{aligned}&x+y+z=5,\\ &x^2+y^2+z^2=9\ ,\\ &(yz,xz,xy)=\lambda_1(1,1,1)+2\lambda_2(x,y,z).\end{aligned}$$

They imply that $yz-xz=2\lambda_2(x-y)$ and $xz-xy=\lambda_2(y-z)$. Either $x=y$ or $x\neq y$ and then $z=-2\lambda_2$. Also either $y=z$ or $y\neq z$ and then $x=-2\lambda_2$. This proves that either $x=y$ or $y=z$ or $x=z$.

Let us assume that $x=y$. We have $2x+z=5$ and $2x^2+z^2=9$. This implies that $0=2x^2+(5-2x)^2-9=6x^2-20x+16=6(x-\frac{5}{3})^2-\frac{50}{3}+16=6(x-\frac{5}{3})^2-\frac{2}{3}$. This equation has two roots: $x=2$ and $x=\frac{4}{3}$. Therefore either $(x,y,z)=(2,2,1)$ or $(x,y,z)=(\frac{4}{3},\frac{4}{3},\frac{7}{3})$. The corresponding values of the function xyz are 4 and $\frac{112}{27}$.

Formally speaking we should consider the possibilities $y=z$ and $x=z$ but it is clear that they lead to points $(2,1,2)$ and $(\frac{4}{3},\frac{7}{3},\frac{4}{3})$ and $(1,2,2)$ and $(\frac{7}{3},\frac{4}{3},\frac{4}{3})$ and the values 4 and $\frac{112}{27}$ for xyz.

It is clear that $4<\frac{112}{27}<\frac{125}{27}$. This proves that

$\inf_M f = 4 = f(2,2,1) = f(2,1,2) = f(1,2,2)$ and $\sup_M f = \frac{125}{27} = f(\frac{5}{3},\frac{5}{3},\frac{5}{3})$. $\square$

2. If $t\geqslant 0$ and if the numbers $x_1,x_2,x_3,\ldots,x_7$ are replaced with the numbers $tx_1,tx_2,tx_3,\ldots,tx_7$ then both sides of the inequality to be proved are multiplied by t. The same is true for y's. If $x_1=x_2=x_3=\cdots=x_7=0$ then the inequality holds (both sides vanish). If at least one of x's is different from 0 then we can multiply them all by such number that $x_1^3+x_2^3+x_3^3+\cdots+x_7^3=1$. In the same way we show that one can assume that $y_1^{3/2}+y_2^{3/2}+y_3^{3/2}+\cdots+y_7^{3/2}=1$. Later on $\mathbf{x}=(x_1,x_2,\ldots,x_7)$ and $\mathbf{y}=(y_1,y_2,\ldots y_7)$. Our aim now is to prove that if these two equations are fulfilled then $x_1y_1+x_2y_2+x_3y_3+\cdots+x_7y_7\leqslant 1$. Let

$g_1(\mathbf{x},\mathbf{y})=x_1^3+x_2^3+x_3^3+\cdots+x_7^3-1$ and

$g_2(\mathbf{x},\mathbf{y})=y_1^{3/2}+y_2^{3/2}+y_3^{3/2}+\cdots+y_7^{3/2}-1$.

The set

$$M=\{(\mathbf{x},\mathbf{y}):\quad g_1(\mathbf{x},\mathbf{y})=0, g_2(\mathbf{x},\mathbf{y})=0, x_1\geqslant 0,\ldots,x_7\geqslant 0, y_1\geqslant 0,\ldots,y_7\geqslant 0\}$$

is compact. Let $f(x_1,x_2,\ldots,x_7,y_1,y_2,\ldots y_7)=x_1y_1+x_2y_2+x_3y_3+\cdots+x_7y_7$. The function f is continuous so it attains $\sup_M f$.

Let $(\mathbf{x},\mathbf{y})$ be a point at an extreme value is attained and $x_i>0, y_i>0$ for all $i\in\{1,2,\ldots,7\}$ Then there are numbers λ_1,λ_2 such that

$$\text{(LE)}\qquad \nabla f(\mathbf{x},\mathbf{y})=\lambda_1\nabla g_1(\mathbf{x},\mathbf{y})+\lambda_2\nabla g_2(\mathbf{x},\mathbf{y}).$$

Clearly

$$\begin{aligned} \nabla f(\mathbf{x},\mathbf{y}) &= (\mathbf{y},\mathbf{x}), \\ \nabla g_1(\mathbf{x},\mathbf{y}) &= 3(x_1^2, x_2^2, \dots, x_7^2, 0, 0, \dots, 0), \\ \nabla g_2(\mathbf{x},\mathbf{y}) &= \tfrac{3}{2}(0, 0, \dots, 0, y_1^{1/2}, y_2^{1/2}, \dots, y_7^{1/2}). \end{aligned}$$

From (LE) and these equations it follows that $y_i = 3\lambda_1 x_i^2$ and $x_i = \frac{3}{2}\lambda_2 y_i^{1/2}$ for all $i \in \{1, 2, \dots, 7\}$ therefore $3\lambda_1 x_i^3 = \frac{3}{2}\lambda_2 y_i^{3/2}$ for all $i \in \{1, 2, \dots, 7\}$. After adding these seven equations we obtain

$$3\lambda_1 = 3\lambda_1(x_1^3 + x_2^3 + \dots + x_7^3) = \tfrac{3}{2}\lambda_2(y_1^{3/2} + y_2^{3/2} + \dots + y_7^{3/2}) = \tfrac{3}{2}\lambda_2.$$

The inequalities $x_i > 0, y_i > 0$ imply that $\lambda_1 > 0$ and $\lambda_2 > 0$. This in turn implies that $x_i^3 = y_i^{3/2}$ so $x_i^2 = y_i$ for all $i \in \{1, 2, \dots, 7\}$. From these seven equations we get

$$x_1y_1 + x_2y_2 + \dots + x_7y_7 = x_1 \cdot x_1^2 + x_2 \cdot x_2^2 + \dots + x_7 \cdot x_7^2 = x_1^3 + x_2^3 + \dots + x_7^3 = 1.$$

We know now that either $\sup_M f = 1$ or $\sup_M f$ is attained at a point with at least one coordinate equal to 0. We can assume that that $x_7 = 0$. We should prove that $x_1y_1 + x_2y_2 + \dots + x_6y_6 \leqslant (x_1^3 + x_2^3 + \dots + x_6^3)^{1/3}(y_1^{3/2} + y_2^{3/2} + \dots + y_6^{3/2} + y_7^{3/2})$. It suffices to prove that

$x_1y_1 + x_2y_2 + \dots + x_6y_6 \leqslant (x_1^3 + x_2^3 + \dots + x_6^3)^{1/3}(y_1^{3/2} + y_2^{3/2} + \dots + y_6^{3/2})$ because $y_1^{3/2} + y_2^{3/2} + \dots + y_6^{3/2} \leqslant y_1^{3/2} + y_2^{3/2} + \dots + y_6^{3/2} + y_7^{3/2}$.

This means that we should prove the same theorem for twelve numbers instead of for fourteen. If we apply the same method we shall see that either the theorem is true or at least one of the twelve n umbers is 0. In this case we shall reduce the problem to the inequality with ten numbers. Then to eight, then to six and then to four numbers that is to the inequality

$$x_1y_1 + x_2y_2 \leqslant (x_1^3 + x_2^3)^{1/3}(y_1^{3/2} + y_2^{3/2})^{2/3}.$$

If one of the numbers x_1, x_2, y_1, y_2 is 0 the inequality becomes obvious. If all four numbers are positive then we apply Lagrange method as we have done at the beginning of the solution. The result is as before. If we assume that $x_1^3 + x_2^3 = 1$ and $y_1^{3/2} + y_2^{3/2} = 1$ then the extreme value can be only 1. This ends the proof. $\square$

Remark. . It is possible to use Lagrange method many times that is to consider three constraints: $x_1^3 + x_2^3 + x_3^3 + \dots + x_7^3 = 1$, $y_1^{3/2} + y_2^{3/2} + y_3^{3/2} + \dots + y_7^{3/2} = 1$ and $x_7 = 0$. Then four constraints and so on. I am convinced that the method presented above gives the result faster.

3. The domain is closed and unbounded. There is no guarantee that there exists a maximal value or a minimal value of the function. We even do not know that the function is bounded from either side. If there is an extreme value and if it is attained at an interior point (x, y) of the domain then $\nabla f(x, y) = (0, 0)$ so $x > 0$ and $0 = \frac{\partial f}{\partial x}(x, y) = \frac{3x^2+y^2+12-6x(x+y)}{(3x^2+y^2+12)^2}$ and $0 = \frac{\partial f}{\partial y}(x, y) = \frac{3x^2+y^2+12-2y(x+y)}{(3x^2+y^2+12)^2}$. This is equivalent to $-3x^2 - 6xy + y^2 + 12 = 0 = 3x^2 - 2xy - y^2 + 12$. Subtract the left hand side from the right hand side: $6x^2 + 4xy - 2y^2 = 0$ therefore

$$0 = 3x^2 + 2xy - y^2 = (x + y)(3x - y).$$

If $y = -x$ then $0 = -3x^2 - 6xy + y^2 + 12 = 4x^2 + 12$, a contradiction. If $y = 3x$ then

$0 = -3x^2 - 6xy + y^2 + 12 = -12x^2 + 12$ so $x = 1$ and $y = 3$ ($x = -1$ is not allowed because we look only at the set A). If $x = 0$ then $f(x, y) = f(0, y) = \frac{y}{y^2+12}$. One can see that $\frac{\partial f}{\partial y}(0, y) = \frac{y^2+12-2y(0+y)}{(y^2+12)^2} = \frac{12-y^2}{(y^2+12)^2}$. This implies immediately that the function $y \mapsto f(0, y)$ decreases on each of the half–lines $(-\infty, -\sqrt{12}]$, $[\sqrt{12}, \infty)$ and increases on the interval $[-\sqrt{12}, \sqrt{12}]$. Since $f(0, y) < 0$ for $y < 0$ and $f(0, y) > 0$ for $y > 0$ we have $-\frac{\sqrt{3}}{12} = -\frac{\sqrt{12}}{24} \leqslant f(0, y) \leqslant \frac{\sqrt{12}}{24} = \frac{\sqrt{3}}{12}$ for each $y \in \mathbb{R}$. This implies that if the maximal value of f in A exists the it equals either $f(0, \sqrt{12}) = \frac{\sqrt{3}}{12}$ or $f(1, 3) = \frac{3+1}{3\cdot 1^2+3^2+12} = \frac{1}{6} > \frac{\sqrt{3}}{12}$, so if $\max_A f$ exist the $\max_A f = \frac{1}{6}$. Analogously if $\min_A f$ exists then $\min_A f = -\frac{\sqrt{3}}{12}$.

The numerator of $f(x, y)$ is a first degree polynomial while its denominator is a quadratic polynomial and $3x^2 + y^2 = 0$ iff $x = y = 0$. Therefore the quadratic should dominate over the first degree expression for (x, y) far away from $(0, 0)$. We are going to show that this is the case. It is well known and easy to prove that $|x + y| \leqslant \sqrt{2(x^2 + y^2)}$ (Cauchy–Schwarz inequality). Thus

$$\left|\frac{x+y}{3x^2+y^2+12}\right| \leqslant \frac{\sqrt{2(x^2+y^2)}}{3x^2+y^2+12} \leqslant \frac{\sqrt{2(x^2+y^2)}}{x^2+y^2} = \frac{\sqrt{2}}{\sqrt{x^2+y^2}}.$$

Therefore if $x^2 + y^2 \geqslant 100$ then $|f(x, y)| = \left|\frac{x+y}{3x^2+y^2+12}\right| \leqslant \frac{\sqrt{2}}{10} < \frac{1}{6}$.

Let $B = \{(x, y): \quad x \geqslant 0, x^2 + y^2 \leqslant 100\}$. This set is bounded and closed. Therefore there exist $\max_B f$ and $\min_B f$. They are attained at points with $x^2 + y^2 < 100$ because $\max_B f \geqslant f(1, 3) = \frac{1}{6}$ and $\min_B f \leqslant f(0, -\sqrt{12}) = -\frac{\sqrt{3}}{12} < -\frac{1}{10}$. This proves that $\min_A f = \min_B f = f(0, -\sqrt{12}) = -\frac{\sqrt{3}}{12}$ and $\max_A f = \max_B f = f(1, 3) = \frac{1}{6}$