Cross-sections of a cube

Let start with some remainders. If $(A,B,C)\neq(0,0,0)$ then the equation $Ax+By+Cz=D$ describes the plane perpendicular to the vector (A,B,C). Let us prove it. Let Π be a set consisting of such points (x,y,z) that $Ax+By+Cz=D$. If $(x_1,y_1,z_1),(x_2,y_2,z_2)\in\Pi$ i.e. if $Ax_1+By_1+Cz_1=D=Ax_2+By_2+Cz_2$ then

$$0=A(x_1-x_2)+B(y_1-y_2)+C(z_1-z_2)=\langle(A,B,C),(x_1-x_2,y_1-y_2,z_1-z_2)\rangle$$

This means that the vectors (A,B,C) and $(x_1-x_2,y_1-y_2,z_1-z_2)$ are perpendicular so if 2 points are in Π then the vector with initial point at one of them and the end point at another one is perpendicular to (A,B,C). On the other hand if $Ax_1+By_1+Cz_1=D$ and $0=Ax+By+Cz$ then $A(x_1+x)+B(y_1+y)+C(z_1+z)=D$. This means that Π consist of all points which are end points of vectors perpendicular to (A,B,C) with initial point (x_1,y_1,z_1).

One may think of equations of planes with $A^2+B^2+C^2=1$ for multiplying an equation by a number different from 0 does not change equation's solutions, in our case it does not change the plane. The result is that the set of vectors under consideration becomes compact (bounded and closed).

Now we are looking at all planes that intersect the unit cube C with the vertices $\mathbf{v}_1=$ $=(0,0,0)$, $\mathbf{v}_2=(1,0,0)$, $\mathbf{v}_3=(1,1,0)$, $\mathbf{v}_4=(0,1,0)$, $\mathbf{v}_5=(0,0,1)$, $\mathbf{v}_6=(1,0,1)$, $\mathbf{v}_7=(1,1,1)$, $\mathbf{v}_8=(0,1,1)$. We want to show that there is a cross-section of C with the largest area. First of all let us notice that if the plane
$Ax+By+Cz=D$ intersects the cube C then

$$|D|=|Ax+By+Cz|\leqslant|Ax|+|By|+|Cz|=|A|x+|B|y+|C|z\leqslant|A|+|B|+|C|.$$

Therefore the set of quadruplets $(A,B,C,D)\in\mathbb{R}^4$ that correspond to the planes that intersect C is bounded. The area of the cross-section depends continuously on the vertices of the polygon. This polygon may a triangle e.g. with the vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, it may be a square or a trapezium e.g. with the vertices $V_1=(\frac12,0,1)$, $V_2=(1,0,0)$, $V_3=(0,\frac23,0)$ and $V_4=(0,\frac13,1)$. Each student should prove that $V_1V_4\parallel V_2V_3$ and $V_1V_2\nparallel V_3V_4$ so the quadrilateral is a trapezium but it is not a parallelogram. Also $\|V_1-V_2\|_2\neq\|V_3-V_4\|_2$ so the trapezium is not equilateral. Write an equation of the plane Π' through the points $(0,0,1)$, $(1,\frac12,0)$ and $(\frac12,1,0)$. How many sides the polygon obtained as a cross–section of C with Π' has?
Now write an equation of the plane Π'' through the points $(\frac12,0,1)$, $(1,\frac12,0)$ and $(\frac12,1,0)$. Show that in this case the cross section is a hexagon. Is it regular (equal sides and equal angles)?

Let V_1,V_2,V_3 be arbitrary points in $\mathbb{R}^3$ and $V_1\neq V_2$. I want to find a point V on the line through V_1,V_2 such that the vector $V_3-V)$ will be perpendicular to the vector V_1-V_2. It is easy. Every point V on the line through V_1,V_2 may be written as $V_1+t(V_2-V_1)$ for some real number t. We want $\langle(V_3-V),(V_2-V_1)\rangle=0$ i.e.

$$0=\langle(V_3-V),(V_2-V_1)\rangle=\langle(V_3-V_1),(V_2-V_1)\rangle-t\,\langle(V_2-V_1),(V_2-V_1)\rangle.$$

Therefore

$$t=\frac{\langle(V_3-V_1),(V_2-V_1)\rangle}{\langle(V_2-V_1),(V_2-V_1)\rangle}.$$

The distance from V_3 to the line through V_1,V_2 equals $\left\|V_3-\left(V_1+\frac{\langle(V_3-V_1),(V_2-V_1)\rangle}{\langle(V_2-V_1),(V_2-V_1)\rangle}\right)(V_2-V_1)\right\|_2$.
Let $\mathbf{u}=V_2-V_1$ and $\mathbf{w}=V_3-V_1$. The distance may be written as $\|\mathbf{w}-\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\mathbf{u}\|_2$. Its square is $\left\langle\mathbf{w}-\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\mathbf{u},\mathbf{w}-\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\mathbf{u}\right\rangle=\langle\mathbf{w},\mathbf{w}\rangle-2\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\langle\mathbf{u},\mathbf{w}\rangle+\left(\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\right)^2\langle\mathbf{u},\mathbf{u}\rangle=\langle\mathbf{w},\mathbf{w}\rangle-\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\langle\mathbf{u},\mathbf{w}\rangle$
This implies that the area of the triangle $V_1V_2V_3$ equals to

$$\tag{1}\tfrac12\|u\|_2\cdot\sqrt{\langle\mathbf{w},\mathbf{w}\rangle-\frac{\langle\mathbf{w},\mathbf{u}\rangle}{\langle\mathbf{u},\mathbf{u}\rangle}\langle\mathbf{u},\mathbf{w}\rangle}=\frac12\sqrt{\langle\mathbf{w},\mathbf{w}\rangle\cdot\langle\mathbf{u},\mathbf{u}\rangle-(\langle\mathbf{w},\mathbf{u}\rangle)^2}.$$

This formula holds in$\mathbb{R}^k$ independently of k we thought of $k=3$ but we never used it in the derivation of the formula. So k can be $2,3,4,\ldots$. The conclusion is that the area is a continuous function of the coordinates of the vertices and it a differentiable function of the coordinates if the point V_3 does not lie on the line through V_1, V_2.

This implies that the area of the cross–section of the cube C depends continuously on the coefficients A, B, C, D defining the plane. Prove it!!! The next conclusion is that there is a cross–section with the largest area (Weierstrass Maximum Principle).

To find this cross–section is another story. What is your guess?

Let us start to work on the largest possible area of a cross-section of the cube. One the the simplest possible sections is one that contains $\mathbf{v}_1$, $\mathbf{v}_5$, $\mathbf{v}_7$ and $\mathbf{v}_3$. Obviously It is a rectangle 2 sides if which have length 1 and 2 other have length $\sqrt{2}$, So the area of it is $\sqrt{2}$.

The cross-section is either a triangle or quadrangle or pentagon or heksagon. There is no other possibility since the cube has 6 faces and the sides of polygon which is a cross-section of the cube are intersections of the plane that is cutting the cube and one ot its faces. So there are at most 6 of them. Let us consider all 4 possibilities.

Triangle . The plane intersects three of the six faces of the cube. No to are parallel because no two sides of a triangle are parallel. It is easy to see that three faces are intersected then they contain one of vertices of the cube. Let it be $(0, 0, 0)$. Let us assume the the three edges that meet at $(0, 0, 0)$. Let us assume that the vertices of the triangle are: $\mathbf{p} = (a, 0, 0)$, $\mathbf{q} = (0, b, 0)$ and $\mathbf{r} = (0, 0, c)$. Let $\mathbf{u} = \mathbf{q} - \mathbf{p} = (-a, b, 0)$ and $\mathbf{w} = \mathbf{r} - \mathbf{p} = (-a, 0, c)$. By formula (1) the area of the triangle is

$$\tfrac{1}{2}\sqrt{\langle \mathbf{w}, \mathbf{w}\rangle \cdot \langle \mathbf{u}, \mathbf{u}\rangle - (\langle \mathbf{w}, \mathbf{u}\rangle)^2} = \tfrac{1}{2}\sqrt{(a^2 + c^2)(a^2 + b^2) - (a^2)^2} = \tfrac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}.$$

Since $a, b, c \in [0, 1]$ this area does not exceed the number $\frac{1}{2}\sqrt{1^21^2 + 1^21^2 + 1^21^2} = \frac{1}{2}\sqrt{3} < \sqrt{2}$. We know now that we have to look for the largest area somewhere else.

I want to reduce using geometry as much as possible in this geometrical problem because I want you to see how the theory works. One theorem which years ago was taught at high schools and according to my knowledge does not appear in the present curricula can be stated as follows.

Theorem 1 (about the area of the projection)
Let Π and P be two nonparallel and nonperpendicular planes in $\mathbb{R}^3$ and let α be the acute angle between them. Let $A \subset \Pi$ be some set with the area $a \in (0, \infty)$ and let B be the orthogonal projection of A onto the plane P with the are b. Then $b = a \cos\alpha$.

I shall sketch the proof of the theorem. First let us notice that it is true if A is a triangle with one side parallel to the line $P \cap \Pi$. Let the length of this side be ℓ and the altitude to this side be h. Then $a = \frac{1}{2}\ell \cdot h$. B is then a triangle with one side ℓ and the corresponding altitude$\ell \cos\alpha$. Therefore $b = \ell \cdot h \cos\alpha = a \cos\alpha$. We proved the theorem in this very special case. Therefore the theorem holds also for sets A which are unions of non–overlapping triangles with one side parallel to $P \cap \Pi$. Therefore it is proved for all polygons because every polygon can be expressed as a union of the triangles with one side parallel to $P \cap \Pi$. To see it you just draw lines parallel to $P \cap \Pi$ through all vertices of the polygon considered. The part of the polygon located between the 2 consecutive lines appears to be the union of triangles or trapezia with some sides parallel to $P \cap \Pi$. Then you divide each trapezium with one of its diagonals and the theorem follows. Other sets one approximates with polygons. This allows us to prove it in the general case. There are some mathematical problems in the above proof but this proofs covers all cases we need. $\square$

Let us mention that the angle between 2 planes is equal to the angle between the vectors perpendicular to these planes. We will be interested in the angle between the plane $z = 0$ and the plane $Ax + By + Cz = D$ that is between the vectors $(0, 0, 1)$ and (A, B, C). Recall that the scalar product of the vectors is $A \cdot 0 + B \cdot 0 + C \cdot 1 = \sqrt{A^2 = B^2 + C^2} \cdot 1 \cdot \cos\alpha$ where α is the angle between the two planes. This implies that

$$\cos\alpha = \frac{C}{\sqrt{A^2 = B^2 + C^2}}\,. \tag{2}$$

Quadrangle. The plane intersects 4 faces so it intersects 2 parallel faces. That is the cross section is a trapezium. Let us assume that these 2 faces are $\mathbf{v}_1\mathbf{v}_4\mathbf{v}_8\mathbf{v}_5$ and $\mathbf{v}_2\mathbf{v}_3\mathbf{v}_7\mathbf{v}_6$.

Case 1. Let us assume that the plane has no common points with the planes $\mathbf{v}_3\mathbf{v}_4\mathbf{v}_8\mathbf{v}_7$ and $\mathbf{v}_5\mathbf{v}_6\mathbf{v}_7\mathbf{v}_8$. It intersects 4 other planes. Therefore it must intersect the edges $\mathbf{v}_1\mathbf{v}_5$, $\mathbf{v}_2\mathbf{v}_6$, $\mathbf{v}_2\mathbf{v}_3$ and $\mathbf{v}_1\mathbf{v}_4$. Let these intersection points be $\mathbf{b}_1 = (0,0,\beta_1)$, $\mathbf{b}_2 = (1,0,\beta_2)$, $\mathbf{b}_3 = (1,\beta_3,0)$ and $\mathbf{b}_4 = (0,\beta_4,0)$. These points lie in a plane $\beta_3(\beta_1-\beta_2)x + \beta_2 y + \beta_3 z = \beta_1\beta_3$ because $\beta_3(\beta_1-\beta_2)\cdot 0 + \beta_2\cdot 0 + \beta_3\cdot\beta_1 = \beta_1\beta_3$ etc. The plane contains $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ so it must contain also $\mathbf{b}_4$ thus $\beta_1\beta_3 = \beta_3(\beta_1-\beta_2)\cdot 0 + \beta_2\cdot\beta_4 + \beta_3\cdot 0 = \beta_2\beta_4$. Thus $\beta_4 = \frac{\beta_1\beta_3}{\beta_2} \in (0,1)$, so $\beta_1\beta_3 < \beta_2$. By the theorem 1 the area of $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$ equals

$$F(\beta_1,\beta_2,\beta_3) = \frac{1}{2}\left(\beta_3 + \frac{\beta_1\beta_3}{\beta_2}\right)\cdot\frac{\sqrt{\beta_3^2(\beta_1-\beta_2)^2 + \beta_2^2 + \beta_3^2}}{\beta_3} = \frac{\beta_1+\beta_2}{2\beta_2}\sqrt{\beta_3^2(\beta_1-\beta_2)^2 + \beta_2^2 + \beta_3^2}.$$

W need to find the least upper bound of F ($i.e.\,\sup(F)$) on its domain. This is equivalent to finding the $\sup(F^2)$. We have $F^2(\beta_1,\beta_2,\beta_3) = \frac{(\beta_1+\beta_2)^2(\beta_3^2(\beta_1-\beta_2)^2+\beta_2^2+\beta_3^2)}{4\beta_2^2}$. This function is increasing in $\beta_3 > 0$.

If $\beta_1 \leqslant \beta_2$ then the largest possible value of β_3 is 1, otherwise it is $\frac{\beta_2}{\beta_1}$. Evaluate
$\frac{\partial(F^2)}{\partial\beta_1}F^2(\beta_1,\beta_2,1) = \frac{2}{4\beta_2^2}\Big((\beta_1+\beta_2)((\beta_1-\beta_2)^2+\beta_2^2+1) + (\beta_1+\beta_2)^2(\beta_1-\beta_2)\Big) =$
$= \frac{\beta_1+\beta_2}{2\beta_2^2}\Big((\beta_1-\beta_2)^2+\beta_2^2+1+(\beta_1+\beta_2)(\beta_1-\beta_2)\Big) = \frac{\beta_1+\beta_2}{2\beta_2^2}((\beta_1-\beta_2)^2+1+\beta_1^2) > 0.$
So the function $F^2(\beta_1,\beta_2,1)$ increases in β_1 thus $F^2(\beta_1,\beta_2,1) \leqslant F^2(\beta_2,\beta_2,1) = \beta_2^2+1 \leqslant 2$.

Now $\beta_1 > \beta_2$. The the largest possible value of β_3 is $\frac{\beta_2}{\beta_1}$. We have

$$F^2\left(\beta_1,\beta_2,\frac{\beta_2}{\beta_1}\right) = \frac{(\beta_1+\beta_2)^2}{4\beta_2^2}\left(\frac{\beta_2^2}{\beta_1^2}(\beta_1-\beta_2)^2+\beta_2^2+\frac{\beta_2^2}{\beta_1^2}\right) = \frac{(\beta_1+\beta_2)^2}{4\beta_1^2}((\beta_1-\beta_2)^2+\beta_1^2+2).$$

In the same way as above we conclude that the function $F^2\left(\beta_1,\beta_2,\frac{\beta_2}{\beta_1}\right)$ increases in β_2 so $F^2\left(\beta_1,\beta_2,\frac{\beta_2}{\beta_1}\right) \leqslant F^2(\beta_1,\beta_1,1) = \beta_1^2+1 \leqslant 2$. Case 1 is completely done.

Case 2. This time we assume that the plane does not intersect the faces $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ and $\mathbf{v}_5\mathbf{v}_6\mathbf{v}_7\mathbf{v}_8$ so in intersects the edges $\mathbf{v}_1\mathbf{v}_5$, $\mathbf{v}_2\mathbf{v}_6$, $\mathbf{v}_3\mathbf{v}_7$ and $\mathbf{v}_4\mathbf{v}_8$. Denote the corresponding intersection points by $b_1 = (0,0,\beta_1)$, $b_2 = (0,0,\beta_2)$, $b_3 = (0,0,\beta_3)$ and $b_4 = (0,0,\beta_4)$. Since the faces are pairwise parallel $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$ is a parallelogram so the diagonals meet at the common midpoint of both of them. The midpoint is $\frac{1}{2}(\mathbf{b}_1+\mathbf{b}_3) = \frac{1}{2}(\mathbf{b}_2+\mathbf{b}_4)$. Therefore $\beta_1+\beta_3 = \beta_2+\beta_4$ or $\beta_3 = \beta_2+\beta_4-\beta_1$. The equation of the plane $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_4$ is $(\beta_1-\beta_2)x + (\beta_1-\beta_4)y + z = \beta_1$. From the theorem 1 it follows that the area of $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4$ equals

$$G(\beta_1,\beta_2,\beta_4) = \sqrt{(\beta_1-\beta_2)^2 + (\beta_1-\beta_4)^2 + 1}.$$

With no loss of generality we may assume that $\beta_1 \leqslant \beta_2 \leqslant \beta_4$ because neither rotation around the line $x = y = \frac{1}{2}$ nor symmetry relative to the plane $x = y$ change the area of the cross-section so we can first turn the cube around the line $x = y = \frac{1}{2}$ by $\pm 90^circ$ or $180°$ so that the lowest of the edges after the rotation will start at the origin and then if necessary apply the symmetry relative to the plane $x = y$. We are looking for the smallest value of G or equivalently for the smallest value of G^2. The arguments have to satisfy the inequalities: $0 \leqslant \beta_1 \leqslant 1$, $0 \leqslant \beta_2 \leqslant 1$, $0 \leqslant \beta_4 \leqslant 1$ and $0 \leqslant \beta_3 = \beta_2+\beta_4-\beta_1 \leqslant 1$ i.e. $\beta_2+\beta_4-1 \leqslant \beta_1 \leqslant \beta_2+\beta_4$. It is clear that that G^2 is decreasing in β_1 (recall that $\beta_1 \leqslant \beta_2 \leqslant \beta 4$). This implies that if $\beta_2+\beta_4-1 < 0$ then $G^2(0,\beta_2,\beta_4) \geqslant G^2(\beta_1,\beta_2,\beta_4)$. We know that $G^2(0,\beta_2,\beta_4) = \beta_2^2+\beta_4^2+1 < (\beta_2+\beta_4)^2+1 < 2$. If $\beta_2+\beta_4-1 \geqslant 0$ then $G^2(\beta_1,\beta_2,\beta_4) \leqslant G^2(\beta_2+\beta_4-1,\beta_2,\beta_4) = (\beta_4-1)^2+(\beta_2-1)^2+1 \leqslant (1-\beta_2-1)^2+(\beta_2-1)^2+1 = \beta_2^2+(\beta_2-1)^2+1$. The biggest value of this quadratic polynomial is attained at the ends of the interval $[0,1]$ (the smallest at $\frac{1}{2}$, but it is out of our interest). This biggest value equals 2.

In this part of the solution no derivatives were used. We did not need them because the investigated functions were quadratics. In this case It is enough to be able to write the function (one or more variables) in a so called canonical form.

Pentagon. In this case the plane intersects five of six faces of the cube. Let us assume that the plane does not meet the square $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$. Therefore it intersects all other faces. It is clear that the plane intersects at least 3 of four of the edges $\mathbf{v}_1\mathbf{v}_5$, $\mathbf{v}_2\mathbf{v}_6$, $\mathbf{v}_3\mathbf{v}_7$, $\mathbf{v}_4\mathbf{v}_8$. With no loss of generality we may assume that it intersects the edge $\mathbf{v}_1\mathbf{v}_5$ at the point $(0,0,\beta_1)$ and that if one of the edges $\mathbf{v}_2\mathbf{v}_6$, $\mathbf{v}_3\mathbf{v}_7$, $\mathbf{v}_4\mathbf{v}_8$ is not intersected by the plane the it is $\mathbf{v}_3\mathbf{v}_7$. Let the intersection points will be $(1,0,\beta_2$ and $(0,1,\beta_4)$. The equation od the plane may be written: $(\beta_1-\beta_2)x+(\beta_1-\beta_4)y+z=\beta_1$. Let us denote $A=\beta_1-\beta_2$, $B=\beta_1-\beta_4$. The equation of the plane takes the form $Ax+By+z=\beta_1$. The area of the orthogonal projection of the cross–section onto the plane $\mathbf{v}_5\mathbf{v}_6\mathbf{v}_7\mathbf{v}_8$ equals $1-\frac{(1+A+B-\beta_1)^2}{2AB}$, so the area of the cross-section is

$$\left(1-\frac{(\beta_1-1-A-B)^2}{2AB}\right)\sqrt{A^2+B^2+1}.$$

The plane intersects the edges at the points $(0,0,\beta_1)$, $(1,0,\beta_1-A)$, $(0,1,\beta_1-B)$, $(1,\frac{\beta_1-A-1}{B},1)$, $(\frac{\beta_1-B-A}{A},1,1)$. This implies that $0<\beta_1<1$, $0<\beta_1-A<1$, $0<\beta_1-B<1$, $0<\frac{\beta_1-A-1}{B}<1$ and $0<\frac{\beta_1-B-1}{A}<1$. This implies that $A<\beta_1<A+1$, $B<\beta_1<B+1$, $A,B<0$, $\beta_1-B-1>A$ i.e. $\beta_1>A+B+1$. Therefore $\left(1-\frac{(\beta_1-1-A-B)^2}{2AB}\right)\sqrt{A^2+B^2+1}<\sqrt{A^2+B^2+1}$. The right hand side is obtained for $1+A+B=\beta_1$. This means that the point $(1,1,1)$ lies on the plane. The result is that 2 vertices of the pentagon coincide so it turns into a quadrilateral. The area grew, so at the end it is less than the area of some quadrilateral so it does not exceed $\sqrt{2}$.

Hexagon. Now the plane intersects all six faces of the cube so it is not parallel to any face. We may write the equation of the plane in the form $Ax+By+z=D$, $A,B,D\in\mathbb{R}$. We may assume that it intersects the following edges $\mathbf{v}_1\mathbf{v}_2$, $\mathbf{v}_2\mathbf{v}_6$, $\mathbf{v}_6\mathbf{v}_7$, $\mathbf{v}_7\mathbf{v}_8$, $\mathbf{v}_8\mathbf{v}_4$ and $\mathbf{v}_4\mathbf{v}_1$. The intersection points are $(\frac{D}{A},0,0)$, $(1,0,D-A)$, $(1,\frac{D-A-1}{B},1)$, $(\frac{D-B-1}{A},1,1)$, $(0,1,D-B)$, $(0,\frac{D}{B},1)$. As in the case of the pentagon one can easily see that the constants A,B,D satisfy many inequalities among them: $A<D<0$, $D<1+A$, $B<D<0$, $D<1+B$ and $A+B+1<D$. The area of the orthogonal projection of the cross–section onto the plane $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ equals $1-\frac{(1+A+B-D)^2}{2AB}-\frac{D^2}{2AB}$, so the area of the cross-section is

$$\left(1-\frac{(1+A+B-D)^2}{2AB}-\frac{D^2}{2AB}\right)\sqrt{A^2+B^2+1}.$$

From the properties of quadratic polynomials it follows that for fixed A,B it attains its biggest value for $D=\frac{1+A+B}{2}$. In this case the plane contains the center of the cube namely the point $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. This biggest value is $\left(1-\frac{(1+A+B)^2}{4AB}\right)\sqrt{A^2+B^2+1}$. The numbers A,B have to satisfy the inequalities: $A<0$, $B<0$, $A+B<-1$ and $|A-B|<1$. The last inequality follows from the following two inequalities $D-1<A<D$ and $D-1<B<D$ which imply that both numbers A,B lie in the same interval of the length 1.

Let us define $u=-(A+B)$ and $v=AB$. We have $u>1$, $1>(A-B)^2=(A+B)^2-2AB=u^2-4v$, so $1>u^2-4v\geqslant 0$. It is easy to see that if the above inequalities are satisfied and $A=-\frac{1}{2}(u+\sqrt{u^2-4v})$ and $B=-\frac{1}{2}(u-\sqrt{u^2-4v})$ the $A+B=-u<-1$ and $|A-B|=\sqrt{u^2-4v}<1$. This means that the the last thing to be done is to find the maximal value or rather $\sup g(u,v)$ where $g(u,v)=\left(1-\frac{(1-u)^2}{4v}\right)\sqrt{1+u^2-2v}$ with (u,v) such that $1>u^2-4v\geqslant 0$ and $u>1$. From these two inequalities it follows that $4v>u^2-1>0$. We shall find $\sup g$ on the set $\overline{G}=\{(u,v):\quad u\geqslant 1,\ 1\geqslant u^2-4v\geqslant 0$ instead of on the set $G=\{(u,v):\quad u>1,\ 1>u^2-4v\geqslant 0$ but this will not change the least upper bound of g due to the continuity of g if we set $g(1,0)=\sqrt{2}$. This extends g to a continuous function on $\overline{G}$ because $\frac{(1-u)^2}{4v} = \frac{(u^2-1)^2}{4v(u+1)^2} \leqslant \frac{16v^2}{4v(u+1)^2} = \frac{v}{(u+1)^2}$ so $\frac{(1-u)^2}{4v} \longrightarrow 0$ if $(u,v) \longrightarrow (1,0)$ in $\overline{G}$. Unfortunately the set $\overline{G}$ is not compact although it is closed because it is unbounded. This means that there is no guarantee that there is a number c such that $\sup g = g(c)$.

Let us notice that if $(u,v) \in G$ then $\frac{u-1}{u+1} = \frac{(u-1)^2}{u^2-1} > \frac{(u-1)^2}{4v} \geqslant \frac{(u-1)^2}{u^2}$ therefore

$$\frac{2}{u+1} = 1 - \frac{u-1}{u+1} < 1 - \frac{(u-1)^2}{4v} \leqslant 1 - \frac{(u-1)^2}{u^2} = \frac{2}{u}\left(2 - \frac{2}{u}\right).$$

Another useful inequality $\sqrt{1+u^2-\frac{1}{2}(u^2-1)} > \sqrt{1+u^2-2v} \geqslant \sqrt{1+u^2-\frac{1}{2}u^2}$ so

$$\sqrt{\frac{1}{2}(u^2+2)} \leqslant \sqrt{1+u^2-2v} < \sqrt{\frac{1}{2}(u^2+3)}.$$

The above inequalities imply that

$$\sqrt{2\frac{u^2+2}{u^2+2u+1}} = \frac{2}{u+1}\sqrt{\tfrac{1}{2}(u^2+2)} < g(u,v) = \left(1 - \frac{(1-u)^2}{4v}\right)\sqrt{1+u^2-2v} <$$
$$< \frac{2}{u}\left(2-\frac{2}{u}\right)\sqrt{\tfrac{1}{2}(u^2+3)} = \left(1 - \frac{1}{2u}\right)\sqrt{2\frac{u^2+3}{u^2}}.$$

This implies that $\lim_{u\to\infty} g(u,v) = \sqrt{2}$ (recall that $u^2-1 \leqslant 4v \leqslant u^2$ so if $u \to \infty$ then $v \to \infty$). Therefore either $\sup g$ is attained at some point or $\sup g = \sqrt{2}$. If there is a point $(x,y) \in G$ so that $\sup g = g(x,y)$ then

$$0 = \frac{\partial g}{\partial u}(x,y) = -\frac{x-1}{2y}\sqrt{1+x^2-2y} + \left(1 - \frac{(x-1)^2}{4y}\right)\frac{x}{\sqrt{1+x^2-2y}},$$

$$0 = \frac{\partial g}{\partial v}(x,y) = \frac{(x-1)^2}{4y^2}\sqrt{1+x^2-2y} - \left(1 - \frac{(x-1)^2}{4y}\right)\frac{1}{\sqrt{1+x^2-2y}}.$$

Therefore $0 = \frac{\partial g}{\partial u}(x,y) + x\frac{\partial g}{\partial v}(x,y) = \left(-\frac{x-1}{2y} + \frac{x(x-1)^2}{4y^2}\right)\sqrt{1+x^2-2y}$. Since $(x,y) \in G$ we have $x > 1$ and $y > 0$ so $1+x^2-2y > 1+x^2-4y \geqslant 0$ therefore $2y = x(x-1)$. Using this equation we can rewrite the equation $0 = \frac{\partial g}{\partial u}(x,y)$ in the following way

$$0 = -\frac{x-1}{x(x-1)}\sqrt{1+x^2-x(x-1)} + \left(1 - \frac{(x-1)^2}{2x(x-1)}\right)\frac{x}{\sqrt{1+x^2-x(x-1)}} = -\frac{\sqrt{x+1}}{x} + \frac{\sqrt{x+1}}{2}.$$

The conclusion is $x = 2$ so $y = 1$. The set G is neither open nor closed. The point $(2,1)$ lies on the boundary of G. There is no critical point in the interior of G. This means that there no hexagon with maximal area. $g(2,1) = \left(1 - \frac{(1-2)^2}{4}\right)\sqrt{1+2^2-2} = \frac{3}{4}\sqrt{3} < \sqrt{2}$ so $g(2,1) \neq \sup g$. We have to take care of the boundary of G. There are three parts of the boundary: $u = 1$, $u^2 = 4v$ and $u^2 - 1 = 4v$.

In the first case we have $g(1,v) = \sqrt{2-2v}$ and $0 = u^2 - 1 \leqslant 4v \leqslant u^2 = 1$. Thus $0 \leqslant g(1,v) \leqslant \sqrt{2}$.

In the second case we have

$$g(u,\tfrac{u^2}{4}) = \left(1 - \frac{(1-u)^2}{u^2}\right)\sqrt{1+u^2-\frac{u^2}{2}} = \frac{2u-1}{u^2}\sqrt{\frac{2+u^2}{2}} = \sqrt{\frac{(2u-1)^2(u^2+2)}{2u^4}} = \sqrt{\tfrac{1}{2}\left(2-\tfrac{1}{u}\right)^2\left(1+\tfrac{2}{u^2}\right)}.$$

It easy to see that the function $t \mapsto \frac{1}{2}(2-t)^2(1+2t^2)$ decreases on the interval $(0,1)$ because its derivative $-(2-t)(1-2t)^2$ is negative on it. Therefore $2 \geqslant \frac{1}{2}(2-t)^2(1+2t^2) \geqslant \frac{3}{2}$. Therefore $\sqrt{\frac{3}{2}} \leqslant g(u,\frac{u^2}{4}) \leqslant \sqrt{2}$.

The third case. If $u > 1$ then

$$g(u,\tfrac{u^2-1}{4}) = \left(1 - \frac{(1-u)^2}{u^2-1}\right)\sqrt{1+u^2-\frac{u^2-1}{2}} = \frac{2u-2}{u^2-1}\sqrt{\frac{3+u^2}{2}} = \frac{2}{u+1}\sqrt{\frac{3+u^2}{2}} = \sqrt{2}\sqrt{\frac{u^2+3}{u^2+2u+1}} < \sqrt{2}.$$

The final conclusion is: the maximal area has the cross–section which contains 2 diagonals of the cube e.g. $\mathbf{v}_1\mathbf{v}_3\mathbf{v}_7\mathbf{v}_5$. Among the cross–section which are pentagons there no with maximal area. The same is true for the hexagons. In these cases it is possible to find a pentagon or hexagon close to the rectangle $\mathbf{v}_1\mathbf{v}_3\mathbf{v}_7\mathbf{v}_5$.