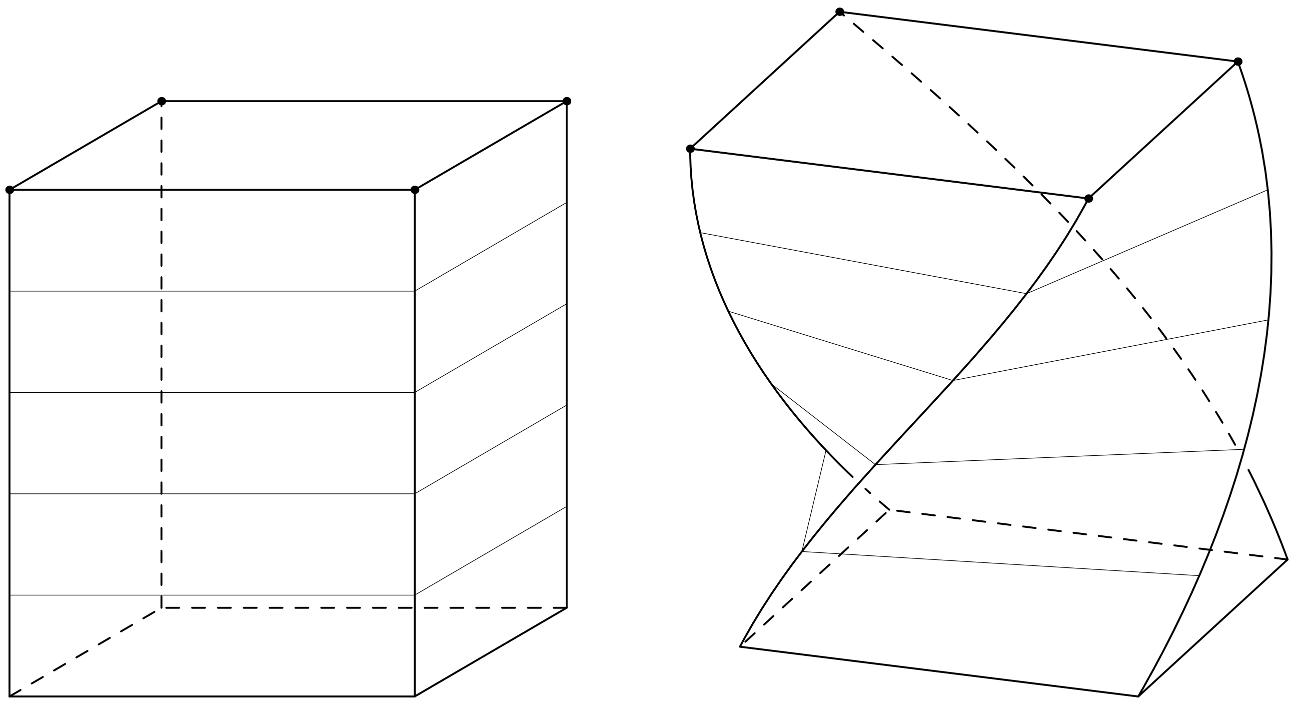
Integration in several variables, part 2

In the first half of the 17-th century the following theorem appeared:

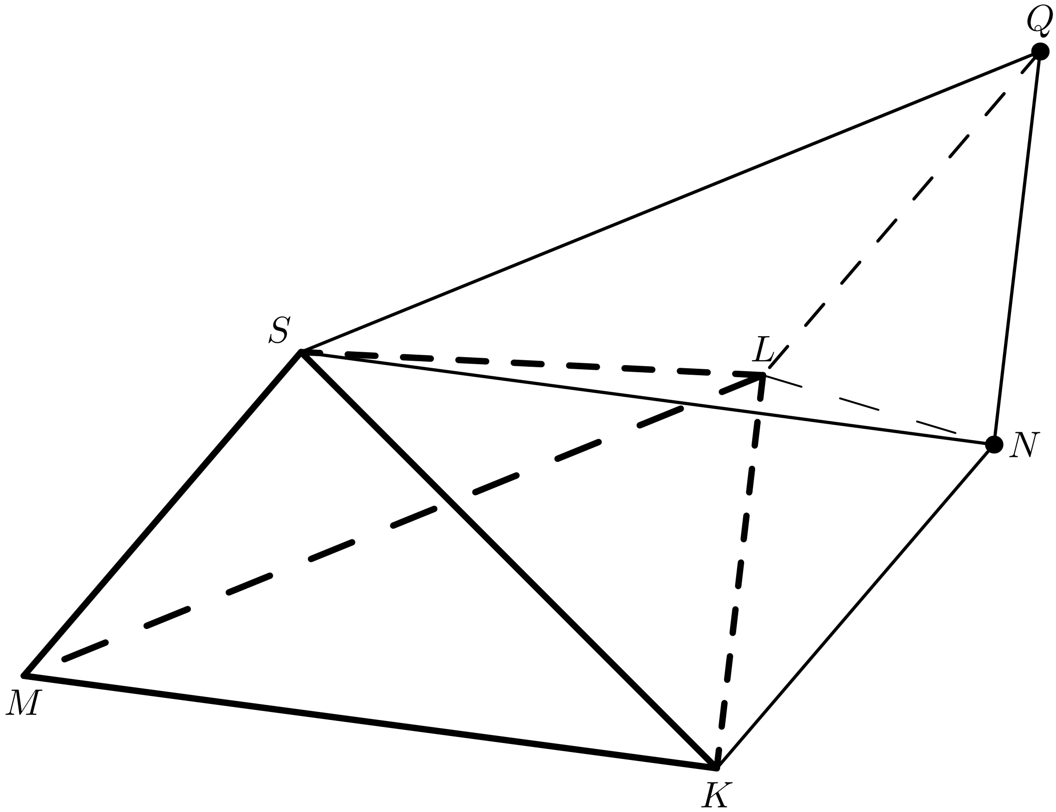
Theorem 11.1 (Cavalieri)
2-dimensional case: Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.

3-dimensional case: Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes. ☐



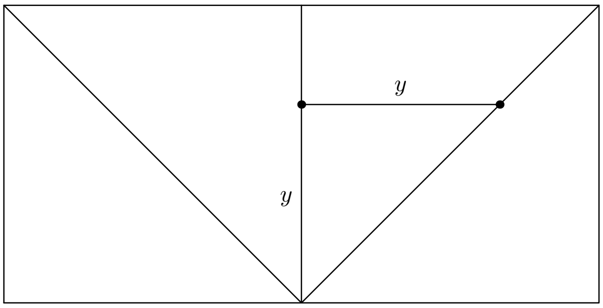
This figure appeared in the textbook for geometry which I used as a high school student (September 1962 – June 1966). At that time it was assumed that the students should be able **to prove** many theorems, not only in Poland. The picture of the figure from the text book is in another file because I cannot overcome some difficulties which appear when I try to include it here. The left box is twisted so that later it looks like the right solid. We assume that the twisting does not change the area of the horizontal cross–sections of the box. Cavalieri theorem ensures us that the volumes of the solids are equal.

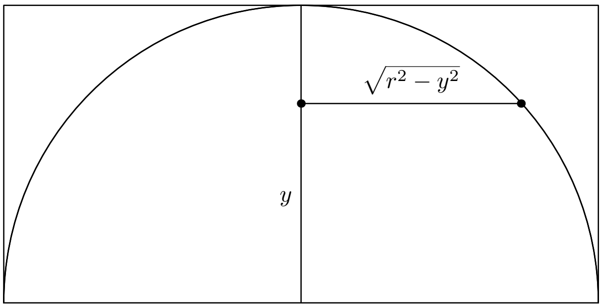
One of the theorems which was proved with the Cavalieri principle was the formula for the volume of the pyramid.



The initial triangular pyramid is $KLMS$ There are two new vertices N, Q such that $MKNS$, $MLQS$ and $KLQN$ are parallelograms. We obtain a triangular prism with six vertices. The edges MS, KN and NQ are parallel. We may think that the prism consists consists of the three triangular pyramids: $KLMS$, $KLNS$ and $NLSQ$. It is not hard to prove with Cavalieri theorem that these pyramids have equal volumes. We can think that the bases of the pyramids $KLMS$ and $KLNS$ are MKS and NSK and that the point L is a common apex of them. The areas of the triangles are equal because they are congruent.From it follows that the same is true for all cross–sections of the two pyramids with the planes parallel to the plane MKS or NKS (different names for the same plane). The same can be said of the pyramids $KLNS$ and $NLSQ$ but this time S is the common apex while the triangles KNL and NQL are bases for the pyramids. They are congruent so they have equal areas and therefore the areas of the cross–sections with a plane parallel to the plane KNL or NQL are also equal. This proves that the volume of each of these pyramids is $\frac{1}{3}$ of the volume of the prism which is the product of the area of the triangle KLM and the distance from the point S to the plane KLM.

Archimedes (287 B.C. – 212 B.C.) was the first to find formulas for many areas and volumes, e.g. the volume of of a ball. Essentially he dealt with the limits and one may say he used the Cavalieri theorem (1635 A.D.). With the Cavalieri theorem it suffices to consider a cylinder of height $r > 0$ and radius r from which a cone of the radius r, the height r and the apex in the center of the base of the cylinder is removed.





This figure shows the vertical cross–sections of the cylinder from which the cone was removed and of the ball. It is easy to see that the horizontal cross–sections have equal areas. In the case of the cylinder we have a disk of radius rv from which the disk of radius y wa removed so the area is $\pi r^2 - \pi y^2$. On the same level in the ball we have the disk of radius $\sqrt{r^2 - y^2}$ so the area is $\pi(r^2 - y^2)$.

This is a short argument but an idea of considering the excavated cylinder was necessary which is obvious only when someone showed it to us.

Nothing important in this respect happened for many years because new way of thinking was necessary and it took people many years to understand it and to discover new methods.

Any proof ot the Cavalieri theorem in dimension three requires in fact dealing with limits of sequences or functions. In the dimension two the situation is different as long as we are interested in areas of polygons only. Anyway the limits are necessary for polyhedrons. This means that also for them integrals are necessary.

It is necessary to say at this moment that integrals make solutions of these problems quite easy and in all cases considered about 2000 years ago the integrals that we have to consider are in fact very simple. You do not need to consider many separate cases for which different ideas are needed.

Now let us solve problem 1b from temat 28. Let us consider a disk D centered at the point $(R, 0, 0)$ of radius $\varrho \in (0, R)$ in the plane $y = 0$. In other words

$$D = \{(x, 0, z) \colon \quad (x - R)^2 + z^2 \leqslant \varrho^2,\ x, z \in \mathbb{R}\}.$$

Then we rotate D around the z–axis. We obtained a solid torus T. T is a three dimensional set, its surface is called torus. We are going to find its volume. In order to do it we need to describe points of T. At first we shall introduce polar coordinates in the disk D. Let $0 < r < \varrho$ and $-\pi < t < \pi$. Let $(x, 0, z) = (R + r\cos t, 0, r\sin t)$. Clearly $(R + r\cos t, 0, r\sin t) \in D$. Since we assumed that $|t| < \pi$ there are points of D which are not written in this way, namely the points $(R - r, 0, 0)$, $0 \leqslant r \leqslant \varrho$. This is one straight line segment. After rotating it around the z–axis we obtain a subset of the plane $(z = 0)$ so its volume is 0. After rotating the point $(R + r\cos t, 0, r\sin t)$ around the z–axis by the angle s we obtain the point

$$\big((R + r\cos t)\cos s, (R + r\cos t)\sin s, r\sin t\big).$$

The map $(r, t, s) \mapsto \big((R + r\cos t)\cos s, (R + r\cos t)\sin s, r\sin t\big)$ is a diffeomorphism of the set $(0, \varrho) \times (-\pi, \pi) \times (-\pi, \pi)$ onto the set $\hat{T}$ that consists of all points of T with two exceptions: the points on the surface of T and some points in the plane $z = 0$ so we excluded a set the volume of which is 0. Let us write the Jacobi matrix of this diffeomorphism:

$$M = \begin{pmatrix} \cos t \cos s & -r \sin t \cos s & -(R + r\cos t)\sin s \\ \cos t \sin s & -r \sin t \sin s & (R + r\cos t)\cos s \\ \sin t & r\cos t & 0 \end{pmatrix}.$$

We can write

$$M^T \cdot M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (R + r\cos t)^2 \end{pmatrix}.$$

It is very well known that $\det(M^T) = \det(M)$ and $\det(M^T \cdot M) = \det(M^T)\cdot\det(M) = \det(M)^2$. Therefore $\det(M)^2 = r^2(R{+}r\cos t)^2$ so $|\det(M)| = r(R{+}r\cos t)$. The volume is therefore equal to $\int_0^{\varrho}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} r(R{+}r\cos t)dsdtdr = 2\pi\int_0^{\varrho}\int_{-\pi}^{\pi} r(R{+}r\cos t)dtdr = 2\pi{\cdot}2\pi R\int_0^{\varrho} rdr = 2\pi{\cdot}2\pi R{\cdot}\frac{1}{2}r^2$. We proved that the volume equals $2\pi^2Rr^2 = 2\pi R \cdot \pi r^2$. It is a product of the distance traveled by the center of the disc and the area of the disc. This is a more general theorem which you do not need to remember.

Theorem 11.2 (Pappus, 290 - 350 A.D., Guldin, 1577 - 1643 A.D.)
If A is a compact set contained in the half-plane $\{(x, y, z)\colon \quad x > 0, y = 0\}$ with a center of mass $C = (x_C, 0, z_C)$ then the volume of the solid of revolution generated by rotating A about the z–axis equals to the product of the area of the set A and the distance traveled by the centroid C $(2\pi_C)$.

The proof easily follows from the change of variables theorem but neither Pappus nor Guldin knew anything of it. We only mention that it was discovered in antiquity then it was forgotten and rediscovered after more than 1200 years. Integrals were defined after both men died ...