1. Is there a C^2 function $f(x,y)$, such that $\frac{\partial f}{\partial x} = k_1 x y^2$ and $\frac{\partial f}{\partial y} = k_2 x^2 y$.

Yes/No: .... 1 pt

Solution. Suppose the function exists. Then $2k_1 xy = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 2k_2 xy$. This implies that $k_1 = k_2$. Therefore if $k_1 \neq k_2$ there is no such function. If $k_1 = k_2$ the function exists e.g. $f(x,y) = \frac{1}{2} k_1 x^2 y^2$. $\square$

Remark 6.1 $\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$ but there exits no such function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ that $\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$ although there exist such functions defined on half–planes the boundaries of which are straight lines through the point $(0,0)$ for example $\arctan \frac{y}{x}$ for $x > 0$ or $\arccos \frac{x}{\sqrt{x^2+y^2}}$ for $y > 0$. Suppose that there is such function f defined for all (x,y) with $x^2 + y^2 > 0$. Then by the chain rule we get $\frac{d}{dt}(f(\cos t, \sin t) = \frac{\partial f}{\partial x}(\cos t, \sin t)(-\sin t) + \frac{\partial f}{\partial \mathbf{y}}(\cos t, \sin t)(\cos t) = \frac{(-\sin t)^2 + (\cos t)^2}{(\cos t)^2 + (\sin t)^2} = 1$. The function $t \mapsto f(\cos t, \sin t)$ is therefore strictly increasing but this is not true because $f(\cos 0, \sin 0) = f(\cos(2\pi), \sin(2\pi))$. The information tells us that the proof without the example of a function at the end would not be complete. The little puncture in the domain (just one point missing) caused serious problems. We end at this place although there is a long story ahead. $\square$

2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x^2 + y^2 + z^2 - \frac{k_1}{k_3}xy + x + 2z + k_2$.
Find and classify the critical points of f.

answer: . . . . 4 pt

Solution. $\nabla f(x,y,z) = (2x - \frac{k_1}{k_3}y + 1, 2y - \frac{k_1}{k_3}x, 2z+2) = (0,0,0)$ iff $z = -1$, $x = \frac{2k_3^2}{k_1^2 - 4k_3^2}$ and $y = \frac{k_1k_3}{k_1^2-4k_3^2}$ – we solved a system of linear equations in x, y, z. Now we need to decide whether or not it is a local maximum, a local minimum or a saddle for f. The following formula explains everything
$f(x,y,z) = \big(y - \frac{k_1}{2k_3}x\big)^2 + \big(1 - (\frac{k_1}{2k_3})^2\big)\big(x + \frac{1}{2(1-(\frac{k_1}{2k_3})^2)}\big)^2 + (z+1)^2 - 1 + k_2 - \frac{1}{4(1-(\frac{k_1}{2k_3})^2)}$.
Squares of real numbers are non–negative, $0 < \frac{k_1}{2k_3} < \frac{1}{2}$ so the function f attains its minimal value for $x = -\frac{1}{2(1-(\frac{k_1}{2k_3})^2} = \frac{2k_3^2}{k_1^2-4k_3^2}$, $y = \frac{k_1}{2k_3}x = \frac{k_1}{2k_3}\frac{2k_3^2}{k_1^2-4k_3^2} = \frac{k_1k_3}{k_1^2-4k_3^2}$ and $z = -1$.
Obviously for $(x,y,z) \neq \big(\frac{2k_3^2}{k_1^2-4k_3^2}, \frac{k_1k_3}{k_1^2-4k_3^2}, -1\big)$ the strict inequality

$$f(x,y,z) > f\big(\tfrac{2k_3^2}{k_1^2-4k_3^2}, \tfrac{k_1k_3}{k_1^2-4k_3^2}, -1\big)$$

holds. In fact at the critical point the function f attains its smallest value so it has absolute (not only local) minimum at the point.
We could have proceeded differently.

$$D^2f(x,y,z) = \begin{pmatrix} 2 & -\frac{k_1}{k_3} & 0 \\ -\frac{k_1}{k_3} & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Obviously $2 > 0$, $\begin{vmatrix} 2 & -\frac{k_1}{k_3} \\ -\frac{k_1}{k_3} & 2 \end{vmatrix} = 4 - (\frac{k_1}{k_3})^2 > 0$, $\begin{vmatrix} 2 & -\frac{k_1}{k_3} & 0 \\ -\frac{k_1}{k_3} & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(4 - (\frac{k_1}{k_3})^2) > 0$ so by the Sylvester criterion the point $\big(\frac{2k_3^2}{k_1^2-4k_3^2}, \frac{k_1k_3}{k_1^2-4k_3^2}, -1\big)$ is local minimum for f □.

Remark 6.2 In fac at the end of the solution we could say more because the second derivative of f is positively defined at all points not only at the critical. This allows us to say that the function f is strictly convex in the whole plane. Using this statement we are allowed to say that the local minimum is in fact global (absolute). This paragraph goes beyond what we were expected to do. It is enough to write that the minimum is local. □

3. Let $g\colon R^3 \to \mathbb{R}$ be given by $g(x,y,z) = xyz + x^2 - 3y^2 + z^3 - 10$ and let $M = \{(x,y,z) \in \mathbb{R}^3\colon \quad g(x,y,z) = 0,\ (x,y,z) \neq (0,0,0)\}$.

Part A. Give the solutions to the equation $\nabla g(x,y,z) = (0,0,0)$.

answer:.................................... 1 pt

Is M a manifold? Yes/No:.................................... 1 pt

Solution. If $\nabla g(x,y,z) = (yz + 2x, xz - 6y, xy + 3z^2) = (0,0,0)$ then $xyz + 2x^2 = 0 = xyz - 6y^2$ so $2x^2 = -6y^2$ but this is possible only for $x = y = 0$. Then $z = 0$, too. This implies that if $(x,y,z) \in M$ then $\nabla g(x,y,z) \neq (0,0,0)$ so M is a manifold. □

Part B. Let $p \in \mathbb{R}^3$ be the point given in case (k_1) of the following list:

(1). $(5,-2,3)$ (2). $(3,0,1)$ (3). $(0,3\sqrt{2},4)$

(4). $(0,-3\sqrt{2},4)$ (5). $(\sqrt{2},0,2)$ (6). $(-\sqrt{2},0,2)$.

Which of the following equations describes the tangent plane to M at the point p.

(1). $-12\sqrt{2}x + 18\sqrt{2}y + 48z = 84$ (2). $-2\sqrt{2}x - 2\sqrt{2}y + 12z = 28$

(3). $12\sqrt{2}x - 18\sqrt{2}y + 48z = 84$ (4). $4x + 27y + 17z = 17$

(5). $2\sqrt{2}x + 2\sqrt{2}y + 12z = 28$ (6). $6x + 3y + 3z = 21$

answer:.................................... 3 pt

Solution. The gradient of the function g at (x,y,z) is perpendicular to the tangent plane at (x,y,z) (in particular the plane given by $Ax + By + Cz = D$ is perpendicular to the vector $\nabla(Ax + By + Cz) = (A,B,C)$) . The following equations are therefore satisfied:

$\nabla g(5,-2,3) = (4,27,17)$ so $4x + 27y + 17z = 20 - 54 + 51 = 17$,

$\nabla g(3,0,1) = (6,3,3)$ so $6x + 3y + 3z = 18 + 0 + 3 = 21$,

$\nabla g(0,3\sqrt{2},4) = (12\sqrt{2},-18\sqrt{2},48)$ so $12\sqrt{2}x - 18\sqrt{2}y + 48z = 0 - 108 + 192 = 84$,

$\nabla g(0,-3\sqrt{2},4) = (-12\sqrt{2},18\sqrt{2},48)$ so $-12\sqrt{2}x + 18\sqrt{2}y + 48z = 0 - 108 + 192 = 84$,

$\nabla g(\sqrt{2},0,2) = (2\sqrt{2},2\sqrt{2},12)$ so $2\sqrt{2}x + 2\sqrt{2}y + 12z = 4 + 0 + 28 = 28$,

$\nabla g(-\sqrt{2},0,2) = (-2\sqrt{2},-2\sqrt{2},12)$ so $-2\sqrt{2}x - 2\sqrt{2}y + 12z = 4 + 0 + 28 = 28$. □

4. Let $F = (F_1, F_2)$ be the function $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ where $F_1(x, y, z) = x + yz - y^3 + z$ and $F_2(x, y, z) = k_1x + k_3y + k_3z - 1$.

Part A. Is the set $M = \{(x, y, z) \in \mathbb{R}^3 \colon F(x, y, z) = (0, 0)\}$ a manifold?

Yes/No: .... 2 pt

Solution. $\nabla F_1(x, y, z) = (1, z - 3y^2, y + 1)$, $\nabla F_2(x, y, z) = (k_1, k_3, k_3)$. These vectors are linearly independent if at least one of the 2×2 determinants below does not vanish

$$\begin{vmatrix} z - 3y^2 & y + 1 \\ k_3 & k_3 \end{vmatrix} = k_3(z - 3y^2 - y - 1), \ (\cdots \neq 0 \Longrightarrow y, z \text{ are locally functions of } x),$$

$$\begin{vmatrix} 1 & y + 1 \\ k_1 & k_3 \end{vmatrix} = k_3 - k_1(y + 1), \ (\cdots \neq 0 \Longrightarrow x, z \text{ are locally functions of } y),$$

$$\begin{vmatrix} 1 & z - 3y^2 \\ k_1 & k_3 \end{vmatrix} = k_3 - k_1(z - 3y^2). \ (\cdots \neq 0 \Longrightarrow x, y \text{ are locally functions of } z).$$

the sentences inside the parenthesis are references to the implicit function theorem.

If the three determinants vanish then

$y = \frac{k_3 - k_1}{k_1}$, $z = \frac{k_3 + 3k_1y^2}{k_1} = \frac{k_3k_1 + 3k_1^2y^2}{k_1^2} = \frac{k_3k_1 + 3(k_1 - k_3)^2}{k_1^2} = \frac{3k_1^2 - 5k_3k_1 + 3k_3^2}{k_1^2}$. This implies that $y + z = \frac{k_3 - k_1}{k_1} + \frac{3k_1^2 - 5k_3k_1 + 3k_3^2}{k_1^2} = \frac{2k_1^2 - 4k_3k_1 + 3k_3^2}{k_1^2}$. The point (x, y, z) should belong to M so by $F_2 = 0$ we get $x = \frac{1}{k_1}\left(1 - \frac{k_3(2k_1^2 - 4k_1k_3 + 3k_3^2)}{k_1^2}\right) = \frac{k_1^2 - 2k_1^2k_3 + 4k_1k_3^2 - 3k_3^3}{k_1^3}$. By $F_1 = 0$ we get $x = y^3 - z(y + 1) = (\frac{k_3 - k_1}{k_1})^3 - \frac{k_3}{k_1} \cdot \frac{3k_1^2 - 5k_3k_1 + 3k_3^2}{k_1^2} = \frac{-2k_3^3 + 2k_3^2k_1 - k_1^3}{k_1^3}$. This is the same number x so it implies that $-2k_3^3 + 2k_3^2k_1 - k_1^3 = k_1^2 - 2k_1^2k_3 + 4k_1k_3^2 - 3k_3^3$ so

$$0 = k_1^2 + k_1^3 - 2k_1^2k_3 + 2k_1k_3^2 - k_3^3 = k_1^2(1 + k_1 - k_3) - k_3(k_1 - k_3)^2.$$

The constants k_1, k_3 were chosen so that $k_1 > 0$ and $1 + k_1 < 1 + k_1 + k_2 = k_3$. Thus the right–hand side is negative, a contradiction. This proves that the gradients ∇F_1, ∇Fx_2 are linearly independent at all points of the set M. Therefore M is a manifold. □

Part B. Let $p = \left(\frac{-1}{k_3 - k_1}, 0, \frac{1}{k_3 - k_1}\right)$. Is the vector $(1, 1, 1)$ an element of the perpendicular space $T_pM^\perp$

Yes/No: .... 2 pt

Solution. The space $T_pM^\perp$ is spanned by the vectors $\nabla F_1(\frac{-1}{k_3 - k_1}, 0, \frac{1}{k_3 - k_1}) = (1, \frac{1}{k_3 - k_1}, 1)$ and $\nabla F_2(\frac{-1}{k_3 - k_1}, 0, \frac{1}{k_3 - k_1}) = (k_1, k_3, k_3)$. So the question is: Do there exist numbers λ_1, λ_2 such that $\lambda_1(1, \frac{1}{k_3 - k_1}, 1) + \lambda_2(k_1, k_3, k_3) = (1, 1, 1)$. Three equations should be satisfied $\lambda_1 + \lambda_2k_1 = 1$, $\frac{\lambda_1}{k_3 - k_1} + \lambda_2k_3 = 1$ and $\lambda_1 + \lambda_2k_3 = 1$. This implies that $\frac{\lambda_1}{k_3 - k_1} = \lambda_1$ so $k_3 = k_1 + 1$ but this does not happen in our situation because $k_3 = k_1 + k_2 + 1 > k_1 + 1$. Therefore $(1, 1, 1) \notin T_pM^\perp$. □

Remark 6.3 We shall remind the implicit function theorem in the situation above.

If F_1 and F_2 are C^1 functions on an open subset of $\mathbb{R}^3$ and $\begin{vmatrix} \frac{\partial F_1}{\partial x}(p, q, r) & \frac{\partial F_1}{\partial y}(p, q, r) \\ \frac{\partial F_2}{\partial x}(p, q, r) & \frac{\partial F_2}{\partial y}(p, q, r) \end{vmatrix} \neq 0$ and $F_1(p, q, r) = 0 = F_2(p, q, r)$ then there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that if $|z - r| < \delta$ then there is a unique pair (x, y) such that $|x - p| < \varepsilon$ and $|y - q| < \varepsilon$ and $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$. The map $z \mapsto (x, y)$ is C^1 on $(r - \delta, r + \delta)$. □

5. Find the maximum and minimum values of the function $f(x,y,z) = k_1x + k_2y$ subject to the constraints:

$$g_1(x,y,z) = (x - \frac{1}{2})^2 + y^2 + z^2 - 1 = 0$$

$$g_2(x,y,z) = (x + \frac{1}{2})^2 + y^2 + z^2 - 1 = 0.$$

answer:....................................4 pt

Solution. Subtract the two equations $0 = (x+\frac{1}{2})^2 - (x-\frac{1}{2})^2 = 2x$ so $x = 0$ under the constraints. We have to find the biggest and the smallest value of k_2y i.e. the biggest and the smallest value of y. Since $y^2 + z^2 = \frac{3}{4}$ we know that $-\frac{\sqrt{3}}{2} \leqslant y \leqslant \frac{\sqrt{3}}{2}$ so for each admissible point (x,y,z) the inequalities

$$-k_2\frac{\sqrt{3}}{2} \leqslant f(x,y,z) \leqslant k_2\frac{\sqrt{3}}{2}$$

are fulfilled so $\max f = k_2\frac{\sqrt{3}}{2}$ and $\min f = -k_2\frac{\sqrt{3}}{2}$ on the admissible set.

Solution 2. Let $M = \{(x,y,z)\colon \quad g_(x,y,z) = 0 = g_2(x,y,z)\}$. Compute the gradients $\nabla g_1(x,y,z) = 2(x - \frac{1}{2}, y, z)$, $\nabla g_2(x,y,z) = 2(x + \frac{1}{2}, y, z)$.
If $(0,0,0) = =\lambda_1\nabla g_1(x,y,z) + \lambda_2\nabla g_2(x,y,z)$ then
$\lambda_1(x - \frac{1}{2}) + \lambda_2(x + \frac{1}{2}) = 0$, $\lambda_1 y + \lambda_2 y = 0$, $\lambda_1 z + \lambda_2 z = 0$.
If $\lambda_1 + \lambda_2 = 0$ then $0 = \lambda_1(x - \frac{1}{2}) + \lambda_2(x + \frac{1}{2}) = x(\lambda_1 + \lambda_2) + \frac{1}{2}(\lambda_2 - \lambda_1) = \frac{1}{2}(\lambda_2 - \lambda_1)$ so $\lambda_1 = \lambda_2 = 0$. If $\lambda_1 + \lambda_2 \neq 0$ then $y = z = 0$ and therefore $(x - \frac{1}{2})^2 = 1$ and $(x + \frac{1}{2})^2 = 1$. The first equation implies that $x = \frac{3}{2}$ or $x = -\frac{1}{2}$. The second equation implies that $x = \frac{1}{2}$ or $x = -\frac{3}{2}$, a contradiction. We proved that the gradients $\nabla g_1(x,y,z)$ and $\nabla g_2(x,y,z)$ are linearly independent at at all points of the set M. Therefore $M \subset \mathbb{R}^3$ is a manifold of dimension $3 - 2 = 1$. M is compact because it is closed and id $(x,y,z) \in M$ then $|y|, |z| \leqslant 1$ and $|x - \frac{1}{2}|, |x + \frac{1}{2}| \leqslant 1$ so $|x| \leqslant \frac{3}{2}$. Therefore $\max_M f$ and $\min_M f$ exist. By the Lagrange theorem at a point at which an extreme value is attained there exist numbers λ_1, λ_2 such that
$(k_1, k_2, 0) = \nabla f(x,y,z) = \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z) =$
$= (\lambda_1(x - \frac{1}{2}) + \lambda_2(x + \frac{1}{2}), y(\lambda_1 + \lambda_2), z(\lambda_1 + \lambda_2)).$
Since $0 < k_2 = y(\lambda_1 + \lambda_2)$ the inequality $\lambda_1 + \lambda_2 \neq 0$ holds. Therefore $0 = z(\lambda_1 + \lambda_2)$ implies $z = 0$. Thus $(x - \frac{1}{2})^2 + y^2 = 1$ and $(x + \frac{1}{2})^2 + y^2 = 1$. Subtract these equations $0 = (x - \frac{1}{2})^2 - (x + \frac{1}{2})^2 = -2x$ so $x = 0$. Therefore $y^2 = \frac{3}{4}$. This shows that there are two candidates for points at which an extreme value is attained: $(0, -\frac{\sqrt{3}}{2}, 0)$ and $(0, \frac{\sqrt{3}}{2}, 0)$. The corresponding values of f are $-k_2\frac{\sqrt{3}}{2}$ and $k_2\frac{\sqrt{3}}{2}$. This proves that $\min_M f = -k_2\frac{\sqrt{3}}{2}$ and $\max_M f = k_2\frac{\sqrt{3}}{2}$. □

6. Let $S \subset \mathbb{R}^3$ be the compact subset bounded by the plane $\{(x,y,z) \in \mathbb{R}^3 : \quad z = -k_1\}$ and the paraboloid $\{(x,y,z) \in \mathbb{R}^3 : \quad z = k_2^2 - x^2 - y^2\}$.

Part A. Express the volume of S as an iterated integral in (x,y,z) coordinates.

answer:.................................... 2 pt

Solution. We have $S = \{(x,y,z) : \quad -k_1 \leqslant z \leqslant k_2^2 - x^2 - y^2\}$. Thus $-k_1 \leqslant k_2^2 - x^2 - y^2$ i.e. $0 \leqslant x^2 + y^2 \leqslant k_1 + k_2^2$. Therefore the volume of S equals

$$V(S) = \int_{-\sqrt{k_1+k_2^2}}^{\sqrt{k_1+k_2^2}} \int_{-\sqrt{k_1+k_2^2-x^2}}^{\sqrt{k_1+k_2^2-x^2}} \int_{-k_1}^{k_2^2-x^2-y^2} dzdydx.$$

We can do it differently.

$$V(S) = \int_{-k_1}^{k_2^2} \int_{-\sqrt{k_2^2-z}}^{\sqrt{k_2^2-z}} \int_{-\sqrt{k_2^2-z-x^2}}^{\sqrt{k_2^2-z-x^2}} dydxdz.$$

Part B. Express the volume of S as an iterated integral in cylindrical coordinates (r, θ, z), i.e., $x = rcos(\theta)$, $y = rsin(\theta)$, $z = z$.

answer:.................................... 2 pt

Solution.

$$V(S) = \int_{-k_1}^{k_2^2} \int_{-\pi}^{\pi} \int_{0}^{\sqrt{k_2^2-z}} rdrd\theta dz \quad \text{or} \quad V(S) = \int_{-\pi}^{\pi} \int_{0}^{\sqrt{k_2^2+k_1}} \int_{-k_1}^{k_2^2-r^2} rdzdrd\theta \qquad \square$$

Part C. What is the volume of S?

answer:.................................... 2 pt

Solution. $V(S) = \int_{-\pi}^{\pi} \int_{0}^{\sqrt{k_2^2+k_1}} \int_{-k_1}^{k_2^2-r^2} rdzdrd\theta = \int_{-\pi}^{\pi} \int_{0}^{\sqrt{k_2^2+k_1}} (k_2^2 + k_1 - r^2) rdzdrd\theta =$
$= 2\pi \left(\frac{1}{2}(k_2^2 + k_1)^2 - \frac{1}{4}(k_2^2 + k_1)^2 \right) = \frac{\pi}{2}(k_2^2 + k_1)^2.$