2. Compute the following integrals

A1. $\int \frac{dx}{x^2(x^2+1)} = \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx = -\frac{1}{x} - \arctan x + C.$

A2. $\int \frac{dx}{x^2(x+1)} = \int \frac{1}{x}\left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \int \left(\frac{1}{x^2} - \frac{1}{x(x+1)}\right) dx = \int \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right) dx =$
$= -\frac{1}{x} - \ln|x| + \ln|x+1| + C.$

A3. $\int \frac{dx}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C.$

A4. $\int \frac{dx}{x(3x^2+5)} = \int \left(\frac{1}{5x} - \frac{3x}{5(3x^2+5)}\right) dx = \frac{1}{5}\ln|x| - \frac{1}{10}\ln(3x^2+5) + C.$

A5. $\int \frac{dx}{x(x+1)(x+2)} = \int \frac{1}{x}\left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)\right) dx =$
$= \int \left(\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}\right)\right) dx = \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C.$

A6. $\int \frac{dx}{x(x^2-1)} = \int \frac{1}{2x}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \int \left(\frac{1}{2(x-1)} - \frac{1}{2x} - \left(\frac{1}{2x} - \frac{1}{2(x+1)}\right)\right) dx =$
$= \int \left(\frac{1}{2(x-1)} - \frac{1}{x} + \frac{1}{2(x+1)}\right)\right) dx = \frac{1}{2}\ln|x-1| - \ln|x| + \frac{1}{2}\ln|x+1| + C.$

B1. $\int_1^2 \frac{\ln(x^2)}{x}dx = \int_1^2 \frac{2\ln x}{x}dx \overset{u=\ln x}{\underset{du=dx/x}{=\!=\!=}} = \int_0^{\ln 2} 2u du = (\ln 2)^2.$

B2. $\int_{\sqrt[3]{e}}^{e} 3x^2\ln(x^3)dx \overset{u=x^3}{\underset{du=3x^2dx}{=\!=\!=}} \int_e^{e^3} \ln u du = (u\ln u - u)\Big|_e^{e^3} =$
$= e^3\ln(e^3) - e^3 - e\ln e + e = 3e^3 - e^3 - e + e = 2e^3.$

B3. $\int_0^{\pi/4} \frac{\cos x}{1-\sin x}dx \overset{u=1-\sin x}{\underset{du=-\cos x dx}{=\!=\!=}} = -\int_1^{1-\sqrt{2}/2} \frac{1}{u}du = -\left(\ln\left(1-\frac{\sqrt{2}}{2}\right) - \ln 1\right) = \ln\frac{2}{2-\sqrt{2}}.$

B4. $\int_{-1}^1 \frac{e^x}{e^{2x}+1}dx \overset{u=e^x}{\underset{du=e^x dx}{=\!=\!=}} \int_{1/e}^e \frac{1}{u^2+1}du = \arctan e - \arctan\frac{1}{e}.$

B5. $\int_{-1}^1 \frac{\sin(x)}{2+\cos(x)}dx \overset{u=2+\cos(x)}{\underset{du=-\sin x dx}{=\!=\!=}} -\int_3^1 \frac{1}{u}du = -\ln 1 + \ln 3 = \ln 3.$

B6. $\int_1^{\sqrt[3]{e}} \frac{\cos(\ln(x^3))}{x}dx \overset{u=\ln x}{\underset{du=1/x}{=\!=\!=}} \int_0^{1/3}\cos(3u)du = \frac{1}{3}\sin(3u)\Big|_0^{1/3} = \frac{\sin 1}{3}.$

C1. $\int_0^1 x\ln x dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_0^1 = -\frac{1}{4} - \lim_{x\to 0^+}\frac{x^2}{4}\ln x = -\frac{1}{4} - \lim_{x\to 0^+}\frac{\ln x}{\frac{4}{x^2}} \overset{de\ l'H}{=\!=\!=}$
$= -\frac{1}{4} - \lim_{x\to 0^+}\frac{1/x}{-8/x^3} = -\frac{1}{4} + \lim_{x\to 0^+}\frac{x^2}{8} = -\frac{1}{4}.$

C2. $\int_0^\infty \frac{e^{-x^{1/3}}}{x^{2/3}}dx \overset{u=x^{1/3}}{\underset{du=\frac{1}{3}x^{-2/3}dx}{=\!=\!=}} \int_0^\infty 3e^{-u}du = -3e^{-u}\Big|_0^\infty = -3\lim_{u\to\infty}e^{-u} + 3 = 3.$

C3. $\int_0^\infty xe^{-x}dx = (-xe^{-x} - e^{-x})\Big|_0^\infty = \lim_{x\to\infty}(-xe^{-x} - e^{-x}) + 1 = 1.$

C4. $\int_0^2 \frac{1}{(x-2)^{2/3}}dx = 3(x-2)^{1/3}\Big|_0^2 = 3\cdot 2^{1/3}$

C5. $\int_4^{36} \frac{1}{(x-4)^{1/5}}dx = \frac{5}{4}(x-4)^{4/5}\Big|_4^{36} = \frac{5}{4}\cdot 32^{4/5} = 20.$

C6. $\int_{-\infty}^0 x^2e^x dx = (x^2-2x+2)e^x\Big|_{-\infty}^0 = 2 - \lim_{x\to-\infty}(x^2-2x+2)e^x = 2.$ □

3. Let S be the set (k_1) in the following list

(1). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} < z,\ x^2+y^2+z^2 < 1\}$
(2). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} \leqslant z,\ x^2+y^2+z^2 < 1\}$
(3). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} \geqslant z,\ x^2+y^2+z^2 < 1\}$
(4). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} > z,\ x^2+y^2+z^2 < 1\}$
(5). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} < z,\ x^2+y^2+z^2 \leqslant 1\}$
(6). $\{(x,y,z)\colon \quad \sqrt{x^2+y^2} \leqslant z,\ x^2+y^2+z^2 \leqslant 1\}$

Is your set S

open?	**Yes/No** :	....	1 pt
closed?	**Yes/No** :	....	1 pt
bounded?	**Yes/No** :	....	1 pt
compact?	**Yes/No** :	....	1 pt
connected?	**Yes/No** :	....	1 pt
convex?	**Yes/No** :	....	1 pt

In the sequel we assume that the z–axis is vertical and directed upwards so x–axis and y–axis are horizontal.

Let us start with identifying the set $C = \{(x,y,z)\colon \quad \sqrt{x^2+y^2} = z\}$. The number $\sqrt{x^2+y^2}$ is the distance from the point (x,y,z) to the line $x = 0 = y$ or equivalently to the point $(0,0,z)$. Since $\sqrt{x^2+y^2} = z$ the number z is nonnegative and it is the distance from the point (x,y,z) to the plane $z = 0$. This means that the set C is is a surface of an infinite cone which one obtains by rotating the half–line $\{(x,0,x)\colon \quad x \geqslant 0\}$ around the z–axis. The set $\{(x,y,z)\colon \quad \sqrt{x^2+y^2+z^2} < z\}$ consists of points that lie above the surface C. The set $\{(x,y,z)\colon \quad \sqrt{x^2+y^2+z^2} > z\}$ consists of points that lie below the surface C.

The set $\Sigma = (x,y,z)\colon \quad \{x^2+y^2+z^2 = 1\}$ is a sphere of the radius 1 centered at the point $(0,0,0)$. The set $\{(x,y,z)\colon \quad x^2+y^2+z^2 \leqslant 1\}$ is the ball of the radius 1 centered at the point $(0,0,0)$. The set $\{(x,y,z)\colon \quad x^2+y^2+z^2 < 1\}$ is the open ball of the radius 1 centered at the point $(0,0,0)$, the points of the sphere are not included into it. The set $\{(x,y,z)\colon \quad x^2+y^2+z^2 > 1\}$ consists of all points that lie outside of the closed ball of the radius 1 centered at the point $(0,0,0)$.

All considered sets are contained in the ball $\{(x,y,z)\colon \quad x^2+y^2+z^2 \leqslant 1\}$. This implies that they are all bounded.

The sets (1) and (4) are defined with strict inequalities so they are open. It is so because small change of the three numbers x,y,z does not change the inequality.

The set (6) is closed as defined by the two non-strict inequalities: if (x_n,y_n,z_n) is in S in this case and $\lim\limits_{n\to\infty}(x_n,y_n,z_n) = (x,y,z)$ then $x^2+y^2+z^2 \leqslant 1$ and $\sqrt{x^2+y^2} \leqslant z$. This follows from basic theorems about the limits of the sequences of real numbers.

The remaining three sets are neither open nor closed. The point $\mathbf{p} = (0.3, 0.4, 0.5)$ is an element of second and the third set but at any ball centered at $\mathbf{p}$ there are points from S and points from outside of S. The point $\mathbf{q} = (0,0,1)$ is in the fifth set S but at any ball centered at $\mathbf{p}$ there are points from S (just move down a little bit) and points from outside of S (just move up a little bit) so this set is neither open nor closed.

The sets (1), (2), (5), (6) are convex because a straight line segment joining arbitrary 2 points of the set is contained in the set. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S$ then the straight line segment joining them consists of points of the form $(x_1, y_1, z_1)+t\big((x_2, y_2, z_2)-(x_1, y_1, z_1)\big)$ with $0 \leqslant t \leqslant 1$. For each $t \in [0,1]$ the number $x_1 + t(x_2 - x_1)$ lies between x_1 and x_2, the same is true for the other two coordinates so the inequalities hold also for $(x_1, y_1, z_1)+t\big((x_2, y_2, z_2)-(x_1, y_1, z_1)\big)$ which means that it is in S. The points $(\pm\frac{1}{2}, 0, 0.1)$ are in S in cases 3 and 4 but midpoint of the segment joining them which is $(0, 0, 0.1)$ does not belong to S.

The sets (1), (2), (5), (6) are connected because they are convex. The sets 3 and 4 are also connected although they are not convex. This can be proved by showing a path contained in S consisting of straight line segments joining the two points if S. This can done as follows. If z–coordinates of the two points are negative then the segment joining the is contained in S. If z–coordinate of one point is 0 and the other one is negative then z–coordinates of all points of the straight line segment that joins the are negative except for one end so the segment is contained in S. If both points lie on the plane $z = 0$ then they can be joined with a path contained in S consisting od two segments or one segment because the only point in S on the plane $z = 0$ (case (3)) is $(0, 0, 0)$. In the case (4) one segment suffices because the point $(0, 0, 0)$ is not in S. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S$ and $z_1, z_2 > 0$ then we move from (x_1, y_1, z_1) to $(x_1, y_1, -z_1)$ then to $((x_2, y_2, -z_2)$ and then to (x_2, y_2, z_2) and we are done.

The set (6) is closed and bounded so it is compact. None of the other five sets is closed so neither of them is compact. □

Remark 6.1 The following theorem is true: *an open subset C of $\mathbb{R}^k$ is connected if and only if for every two points $\mathbf{p}, \mathbf{q} \in C$ there exist o polygonal chain contained in C connecting $\mathbf{p}$ with $\mathbf{q}$.* This theorem is not true in case of other sets e.g. a sphere contains no straight line segment although it is connected, the set $\{(x, \sin\frac{1}{x}) : \quad x \in (0, 1]\} \cup \{(0, y) : \quad |y| \leqslant 1\}$ is compact and connected but there is no path contained in C connecting $(0, 0)$ with $(\frac{1}{\pi}, 0)$, this shows that replacing polygonal chain with more general path does not improve the situation. Good news for students is that the economists do not have to deal with such strange sets. The proof of the theorem stated above is not difficult. □

4. Give the value of the limit or state that it does not exist:

1). $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2+y^2}$, 2). $\lim\limits_{(x,y)\to(0,0)} \frac{x^2\cdot y^2}{x^4+2y^4}$, 3). $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$,

4). $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4}$, 5). $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$, 6). $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

1). $0 \leqslant |\sin(x^2y)| \leqslant x^2|y| \leqslant (\sqrt{x^2+y^2})^{2+1}$ so $\frac{\sin(x^2y)}{x^2+y^2} \leqslant \frac{\sqrt{x^2+y^2}^3}{x^2+y^2} = \sqrt{x^2+y^2}$. This proves that $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2+y^2} = 0$.

2). The limit does not exist because $\lim\limits_{x\to0} \frac{x^2\cdot x^2}{x^4+2x^4} = \frac{1}{3} \neq \frac{4}{18} = \lim\limits_{x\to0} \frac{(2x)^2\cdot x^2}{(2x)^4+2x^4}$.

3). The limit does not exist because $\lim\limits_{x\to0} \frac{x\cdot x^2}{x^3+x^3} = \frac{1}{2} \neq \frac{2}{9} = \lim\limits_{y\to0} \frac{(2y)y^2}{(2y)^3+y^3}$.

4). $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4} = 0$ because $|x^3y^2| \leqslant (\sqrt{x^2+y^2})^{3+2}$ and $2(x^4+y^4) \geqslant (x^2+y^2)^2$ which is equivalent to $(x^2-y^2)^2 \geqslant 0$ so $\left|\frac{x^3y^2}{x^4+y^4}\right| \leqslant \frac{(\sqrt{x^2+y^2})^5}{\frac{1}{2}(x^2+y^2)^2} = 2\sqrt{x^2+y^2}$.

5). $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$ because $|xy| \leqslant \sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}$ so $\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leqslant \sqrt{x^2+y^2}$.

6). $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$ because $\frac{xy^2}{x^2+y^2} \leqslant \frac{\sqrt{x^2+y^2}(x^2+y^2)}{x^2+y^2} = \sqrt{x^2+y^2}$. □

Remark 6.2 The ratio $\frac{xy^2}{x^3+y^3}$ assumes arbitrarily big values at points (x,y) arbitrarily close to 0. To understand it it is enough to look at the points $(x,-x+x^2)$. Then $\frac{x(-x+x^2)^2}{x^3+(-x+x^2)^3} = = \frac{x^3(-1+x)^2}{3x^4-3x^5+x^6} = \frac{(-1+x)^2}{x(3-3x+x^2)}$. And $\lim\limits_{x\to 0^\pm} \frac{(-1+x)^2}{x(3-3x+x^2)} = \pm\infty$. The point $(x,-x+x^2)$ is much closer to the line $y=-x$ than to the point $(0,0)$ when x is small in absolute value. □

5. Let $h(x,y)$ denote the function from your question 4. Let A denote the set on which the denominator in h is nonzero. Let $f\colon A\to\mathbb{R}$ be given by

$$f(x,y) = \begin{cases} h(x,y) & \text{for} \quad (x,y)\in A, \\ 0 & \text{for} \quad (x,y) = (0,0). \end{cases}$$

Is f continuous at the point $(0,0)$? **yes/no:** .... 1 pt

Is f continuous at the point $(1,1)$? **yes/no:** .... 1 pt

If h does not have a limit at $(0,0)$ the it is not continuous, if the limit exists and equals to 0 then h is continuous at $(0,0)$. This follows from the definition of the continuity. At $(1,1)$ each of the functions under consideration is a quotient of two continuous functions different from 0 at $(1,1)$ so it is continuous at $(1,1)$. □

6. Calculate the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ for the function

$$f\colon \{(x,y,z)\in\mathbb{R}^3:\quad x\neq 0\} \longrightarrow \mathbb{R}$$

given in (k_3) below

$f(x,y,z) = \frac{\sin(yz)}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{\sin(yz)}{x^2}$, $\quad \frac{\partial f}{\partial y} = \frac{z\cos(yz)}{x}$, $\quad \frac{\partial f}{\partial z} = \frac{y\cos(yz)}{x}$.

$f(x,y,z) = \frac{e^{yz}}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{e^{yz}}{x^2}$, $\quad \frac{\partial f}{\partial y} = \frac{ze^{yz}}{x}$, $\quad \frac{\partial f}{\partial z} = \frac{ye^{yz}}{x}$.

$f(x,y,z) = \frac{\arctan(yz)}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{\arctan(yz)}{x^2}$, $\quad \frac{\partial f}{\partial y} = \frac{z}{x(1+y^2z^2)}$, $\quad \frac{\partial f}{\partial z} = \frac{y}{x(1+y^2z^2)}$.

$f(x,y,z) = \frac{\ln(y^2+z^2)}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{\ln(y^2+z^2)}{x^2}$, $\quad \frac{\partial f}{\partial y} = \frac{2y}{x(y^2+z^2)}$, $\quad \frac{\partial f}{\partial z} = \frac{2z}{x(y^2+z^2)}$.

$f(x,y,z) = \frac{\cos(yz)}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{\cos(yz)}{x^2}$, $\quad \frac{\partial f}{\partial y} = -\frac{z\sin(yz)}{x}$, $\quad \frac{\partial f}{\partial z} = -\frac{y\sin(yz)}{x}$.

$f(x,y,z) = \frac{\arctan(y+z)}{x}$: $\quad \frac{\partial f}{\partial x} = -\frac{\arctan(y+z)}{x^2}$, $\quad \frac{\partial f}{\partial y} = \frac{1}{x(1+(y+z)^2)}$, $\quad \frac{\partial f}{\partial z} = \frac{1}{x(1+(y+z)^2)}$.

The solution is finished. □

7. Let f be your function from question 6. Calculate the directional derivative $\frac{\partial f}{\partial V}(1,1,1)$ where $V=(1,1,2)$. (Your answer should contain expressions such as $\cos(3)$, $\ln(9)$, $\arctan(6)$, etc, that is do not give decimal answers.)

1. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-\sin 1,\cos 1,\cos 1)\cdot(1,1,2) = -\sin 1 + 3\cos 1$;
2. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-e,e,e)\cdot(1,1,2) = -e+3e = 2e$;
3. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-\frac{\pi}{4},\frac{1}{2},\frac{1}{2})\cdot(1,1,2) = -\frac{\pi}{4}+\frac{1}{2}+1 = \frac{3}{2}-\frac{\pi}{4}$.
4. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-\ln 2,1,1)\cdot(1,1,2) = -\ln 2+1+2 = 3-\ln 2$.
5. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-\cos 1,-\sin 1,-\sin 1)\cdot(1,1,2) = -\cos 1 - 3\sin 1$.
6. $\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1)\cdot(1,1,2) = (-\arctan 2,\frac{1}{5},\frac{1}{5})\cdot(1,1,2) = \frac{3}{5}-2\arctan 2$. □

8. Let f be your function from question 6. Which vector $V \in \mathbb{R}^3$ of length 1 maximizes the directional derivative $\frac{\partial f}{\partial V}(1,1,1)$? (Your answer should contain expressions such as $\cos(3)$, $\ln(9)$, $\arctan(6)$, etc, that is do not give decimal answers.)

Let us recall that if $V = (v_1, v_2, v_3)$ and $\nabla f(1,1,1) = (a,b,c)$ then
$\frac{\partial f}{\partial V}(1,1,1) = \nabla f(1,1,1) \cdot (v_1, v_2, v_3) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (v_1, v_2, v_3) = (a,b,c) \cdot (v_1, v_2, v_3) = \leqslant$
$\leqslant \sqrt{a^2+b^2+c^2} \cdot \sqrt{v_1^2+v_2^2+v_3^2} = \sqrt{a^2+b^2+c^2}.$

We have $(a^2+b^2+c^2)\cdot(v_1^2+v_2^2+v_3^2) - (av_1+bv_2+cv_3)^2 = a^2v_2^2 + a^2v_3^2 + b^2v_1^2 + b^2v_3^2 +$
$+ c^2v_1^2 + c^2v_2^2 - 2abv_1v_2 - 2acv_1v_3 - 2bcv_2v_3 = (av_2 - bv_1)^2 + (av_3 - cv_1)^2 + (bv_3 - cv_2)^2.$
This sum equals 0 iff $av_2 = bv_1$, $av_3 = cv_1$ and $bv_3 = cv_2$. This implies that

$$a^2(v_1^2+v_2^2+v_3^2) = v_1^2(a^2+b^2+c^2).$$

Since $v_1^2+v_2^2+v_3^2 = 1$ and $a^2+b^2+c^2 > 0$ we can write $v_1 = r\frac{a}{\sqrt{a^2+b^2+c^2}}$ with $r \in \{-1,1\}$. In the same way we show that $v_2 = s\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $v_3 = t\frac{c}{\sqrt{a^2+b^2+c^2}}$ for some $s,t \in \{-1,1\}$. This implies that

$$\frac{sab}{\sqrt{a^2+b^2+c^2}} = \frac{rab}{\sqrt{a^2+b^2+c^2}}, \qquad \frac{tac}{\sqrt{a^2+b^2+c^2}} = \frac{rac}{\sqrt{a^2+b^2+c^2}}, \qquad \frac{tbc}{\sqrt{a^2+b^2+c^2}} = \frac{sbc}{\sqrt{a^2+b^2+c^2}}.$$

The numbers ab, bc, ca are different from 0. Thus $r = s = t$ so either $V = \frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$ or $V = -\frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$. This proves that the maximal value of $\frac{\partial f}{\partial V}(1,1,1) = av_1 + bv_2 + cv_3$ is attained for $V = \frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$ and it equals $\sqrt{a^2+b^2+c^2}$. We proved that $V = \frac{\nabla f(1,1,1)}{\|\nabla f(1,1,1)\|}$. The answers are:

1. $\left(\frac{-\sin 1}{\sqrt{1+\cos^2 1}}, \frac{\cos 1}{\sqrt{1+\cos^2 1}}, \frac{\cos 1}{\sqrt{1+\cos^2 1}}\right)$ 2. $\left(-1\frac{1}{\sqrt{3}}, 1\frac{1}{\sqrt{3}}, 1\frac{1}{\sqrt{3}}\right)$
3. $\left(\frac{-\pi}{\sqrt{\pi^2+8}}, \frac{2}{\sqrt{\pi^2+8}}, \frac{2}{\sqrt{\pi^2+8}}\right)$ 4. $\left(\frac{-\ln 2}{\sqrt{(\ln 2)^2+2}}, \frac{1}{\sqrt{(\ln 2)^2+2}}, \frac{1}{\sqrt{(\ln 2)^2+2}}\right)$ □
5. $\left(\frac{-\cos 1}{\sqrt{1+\sin^2 1}}, \frac{-\sin 1}{\sqrt{1+\sin^2 1}}, \frac{-\sin 1}{\sqrt{1+\sin^2 1}}\right)$ 6. $\left(\frac{-5\arctan 2}{25(\arctan 2)^2+2}, \frac{1}{25(\arctan 2)^2+2}, \frac{1}{25(\arctan 2)^2+2}\right)$

Remark 6.3 In fact the above proof was unnecessary because it is in prof. Warhurst's notes. In short it says: the gradient shows the direction at which the function grows the fastest. We have to talk of the unit vectors because if the vector is multiplied by a number the the corresponding directional derivative is multiplied by the same number. This cannot be done in when only unit vectors are considered. We pick up just one vector along which the directional derivative is the biggest. This one is uniquely defined at each point at which the gradient does not vanish. In the proof we used the property $a \neq 0$, $b \neq 0$ and $c \neq 0$. This is not necessary. Is just shortens the proof by two or three lines. The students who do not remember the proof should complete it i.e. prove the theorem in the remaining case: one of the numbers a, b, c is 0 and two of the numbers a, b, c are zeros. It is easy! We should also stress that the result proved above shows that the vector $V = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$ is the only one for which the maximum is attained. If we wanted just to show that there is such vector we would check that for this one the inequality turns into equality. □

9. Let $A=\{(x,y)\colon\ \frac{x^2}{4}+\frac{y^2}{9}\leqslant 1\}$ and let $f\colon A\longrightarrow\mathbb{R}$ be case (k_1) below

(1) $f(x,y)=3x^3+2y^2$ (2) $f(x,y)=x^5-2y^2$ (3) $f(x,y)=x^2+y^3$

(4) $f(x,y)=27x^2+2y^3$ (5) $f(x,y)=6x^2+y^3$ (6) $f(x,y)=45x^2+2y^5$.

What are the critical points of f?

What are the maximum and minimum values of f on the boundary of A?

What are the maximum and minimum values of f on A?

Each question is worth 1 point.

The set A is compact. All considered functions are continuous on A. By Weierstrass maximum principle each of them attains its least upper bound and greatest lower bound. The point at which an extreme value is attained may lie inside of the domain and then it must be a critical point of the function or it may be a boundary point not necessarily critical.

In all cases the only critical point is $(0,0)$ — straightforward calculation. The value of each function at the origin is 0.

It is neither the biggest value of the function nor the smallest one: in the first and in the second case we look at $f(x,0)$. This is $3x^3$ or x^5 so it is positive at many points and negative at many other points. At all other cases we look at the function $f(0,y)$ and as in the previous case it is odd degree monomial so its values are positive at many points and negative at many others.

This means that in all cases maximal and minimal values are attained at some boundary points so the answers to the second and to the third questions coincide. We shall look at the boundary points only i.e. $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(1) $f(x,y)=3x^3+2y^2=3x^3+2\cdot 9(1-\frac{x^2}{4})=:g(x)$. We have $g'(x)=9x^2-9x$, $-2\leqslant x\leqslant 2$. $g'(x)=0$ iff $x=0$ or $x=1$. The following equalities hold $g(-2)=-24$, $g(0)=18$, $g(1)=3+\frac{27}{2}=\frac{33}{2}$ and $g(2)=24$. Maximal value of g is therefore $g(2)=24$. The least value of g is -24.

(2) $f(x,y)=x^5-2y^2=x^5-2\cdot 9(1-\frac{x^2}{4})=:g(x)$. We have $g'(x)=5x^4+9x=x(5x^3+9)$, $-2\leqslant x\leqslant 2$. $g'x)=0$ iff $x=0$ or $x=-\sqrt[3]{\frac{9}{5}}$. The following equalities hold $g(-2)=-32$, $g(0)=-18$, $g(2)=32$. We also have $g\left(-\sqrt[3]{\frac{9}{5}}\right)=\left(-\sqrt[3]{\frac{9}{5}}\right)^5-2\cdot 9\left(1-\frac{1}{4}\left(-\sqrt[3]{\frac{9}{5}}\right)^2\right)<0$ and $g\left(-\sqrt[3]{\frac{9}{5}}\right)=-18+\left(-\sqrt[3]{\frac{9}{5}}\right)^2\left(\frac{9}{2}-\frac{9}{5}\right)>-18$. Maximal value of g is therefore $g(2)=32$ and the least value of g is -32.

(3) $f(x,y)=x^2+y^3=4(1-\frac{y^2}{9})+y^3=:g(y)$. We have $g'(y)=-\frac{8}{9}y+3y^2$, $-3\leqslant y\leqslant 3$. $g'(y)=0$ iff $y=0$ or $y=\frac{8}{27}$. We have $g(-3)=-27$, $g(0)=4$ and $g(3)=27$. We have $g\left(\frac{8}{27}\right)=4(1-\frac{1}{9}\cdot\left(\frac{8}{27}\right)^2)+\left(\frac{8}{27}\right)^3=4(1-\left(\frac{8}{81}\right)^2)+\left(\frac{8}{27}\right)^3\in(0,5)$. Therefore the maximal value of g is 27 and its minimal value is -27.

(4) $f(x,y)=27\cdot 4(1-\frac{y^2}{9})+2y^3=:g(y)$. $g'(y)=-24y+6y^2$ for $-3\leqslant y\leqslant 3$. $g'(y)=0$ iff $y=0$ or $y=4>3$. We have $g(-3)=-54$, $g(0)=108$ and $g(3)=54$. So the maximal value of g is 108, the minimal is -54.

(5) $f(x,y)=6x^2+y^3=6\cdot 4(1-\frac{y^2}{9})+y^3=:g(y)$. We have $g'(y)=-\frac{16}{3}y+3y^2$, so $g'(0)=0$ iff $y=0$ or $y=\frac{16}{9}$. The following equalities hold $g(-3)=-27$, $g(0)=24$ and $g(3)=27$. Also $\frac{16}{9}<2$, so $g(\frac{16}{9})=24\left(1-(\frac{16}{27})^2\right)+(\frac{16}{9})^3<24\cdot\frac{3}{4}+8=26<27$, we used

the inequality $\left(\frac{16}{27}\right)^2 > \frac{1}{4}$ equivalent to $\left(\frac{32}{27}\right)^2 > 1$. The maximal value of g is therefore 27 and the minimal value is -27.

(6) $f(x,y) = 45 \cdot 4(1 - \frac{y^2}{9}) + y^5 =: g(y)$. We have $g'(y) = -40y + 5y^4$ so $g'(y) = 0$ iff $y = 0$ or $y = 2$. The following equalities hold: $g(-3) = -243$, $g(0) = 180$, $g(2) = 180 - 80 + 32 = 132$, $g(3) = 243$. This proves that the maximal value of g is 243 while the minimal value is -243. $\square$

Remark 6.4 The functions were chosen so that it was easy to reduce the problem of finding the biggest and the smallest value on the boundary to a one dimensional problem and this one was also easy. One could have a good chance of proceeding in worse way than above. In the third problem one could try to look at a function of x not as we did. Then we were forced to consider two cases $y = \pm 3\sqrt{1 - \frac{x^2}{4}}$ which would lead to more complicated functions then in the solution above. Soon you will learn another method which in many cases will simplify computations. Another important observation was that the set A was compact and this allowed us to use the Weierstrass max/min theorem. We knew that there were points at which the function attained its biggest and its smallest values. In general the function does not need to have the smallest or the biggest value even if it is bounded. Just look at the function $\frac{1}{1+x^2}\sin\frac{1}{x}$ on the interval $(0,1]$. The interval is not compact because 0 is not included into it. The function does not have a limit at 0, so it cannot be extended to a continuous function on $[0,1]$ and therefore the Weierstrass theorem cannot be applied. In all cases in the problem 9 we used it. There was no reason to look at monotonicity of functions of one variable. It sufficed to find at first critical points of the function of two variables and later critical points of a function of one variable. Then look at the values at them and later at a boundary points. Sometimes it is harder. You might try to understand the story about cross sections of a cube: https://www.mimuw.edu.pl/ krych/ekonomia/ek19/cube.pdf and to see that in this case things are more complicated. $\square$

10. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function in case (k_3) below

(1) $f(x,y) = x^3 + y^3 + 6xy + 3$ (2) $f(x,y) = x^3 + y^3 - 6xy + 3$

(3) $f(x,y) = 2x^3 + y^3 - 3x^2 - 3y^2$ (4) $f(x,y) = x^3 + y^3 - 3xy + 3$

(5) $f(x,y) = x^3 + 2y^3 - 3x^2 - 3y^2$ (6) $f(x,y) = x^3 + y^3 + 3xy + 3$

What are the critical points for f ?

What are the values of Hessian determinant $\det D^2 f(x,y)$ at each critical point (x,y) ?

Classify the critical points, i.e., determine if the point gives a saddle, a local max or a local min.

Each question is worth 1 point.

(1) $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 6y$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 + 6x$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$. This implies that $3x^3 + 6xy = x\frac{\partial f}{\partial x} = 0 = y\frac{\partial f}{\partial y} = 3y^3 + 6xy$ thus $x = y$. Therefore $0 = 3x^2 + 6y = 3x^2 + 6x$ so either $x = 0$ or $x = -2$. We proved that there are two critical points: $(0,0)$ and $(-2,-2)$. We have $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 6$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 6y$. The determinants are

$$\begin{vmatrix} 6\cdot 0 & 6 \\ 6 & 6\cdot 0 \end{vmatrix} = -36 \quad \text{and} \quad \begin{vmatrix} 6\cdot(-2) & 6 \\ 6 & 6\cdot(-2) \end{vmatrix} = (-12)^2 - 6^2 = 108.$$

The first one is negative so $(0,0)$ is a saddle for f by Sylvester's criterion. In the second case the determinant is positive and $\frac{\partial^2 f}{\partial x^2}(-2,-2) = -12 < 0$ so by the Sylvester's criterion the point $(-2,-2)$ is o local maximum for f.

(2) $\frac{\partial f}{\partial x}(x,y) = 3x^2 - 6y$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 - 6x$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$. This implies that $3x^3 - 6xy = x\frac{\partial f}{\partial x} = 0 = y\frac{\partial f}{\partial y} = 3y^3 - 6xy$ thus $x = y$. Therefore $0 = 3x^2 - 6y = 3x^2 - 6x$ so either $x = 0$ or $x = 2$. We proved that there are two critical points: $(0,0)$ and $(2,2)$. We have $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = -6$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 6y$. The determinants are

$$\begin{vmatrix} 6\cdot 0 & -6 \\ -6 & 6\cdot 0 \end{vmatrix} = -36 \quad \text{and} \quad \begin{vmatrix} 6\cdot 2 & -6 \\ -6 & 6\cdot 2 \end{vmatrix} = 12^2 - (-6)^2 = 108.$$

The first one is negative so $(0,0)$ is a saddle for f by Sylvester's criterion. In the second case the determinant is positive and $\frac{\partial^2 f}{\partial x^2}(-2,-2) = 12 > 0$ so by the Sylvester's criterion the point $(-2,-2)$ is o local minimum for f.

(3) $\frac{\partial f}{\partial x}(x,y) = 6x^2 - 6x$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 - 6y$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$ i.e. $6x^2 - 6x = 0 = 3y^2 - 6y$. This implies that $x = 0$ or $x = 1$ and $y = 0$ or $y = 2$. We have four critical points: $(0,0)$, $(0,2)$, $(1,0)$ and $(1,2)$. The matrices $D^2 f$ for them are

$$\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

Therefore the Hessian determinants are 36, -36, -36 and 36. This implies that $(0,0)$ is a local maximimum for f, $(0,2)$ and $(1,0)$ are saddles and $(1,2)$ is a local minimum for f by Silvester's criterion.

(4) $\frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 - 3x$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$ i.e. $3x^2 - 3y = 0 = 3y^2 - 3x$. This implies that $x^3 = xy = y^3$ thus $x = y$. Therefore $0 = x^3 - y = x^2 - x$ hence $x = 0$ or $x = 1$. There are two critical points for f: $(0,0)$ and $(1,1)$. The matrices $D^2 f$ for them are $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ and $\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$. The Hessian determinants are -9 and 27. Therefore $(0,0)$ is a saddle for f and $(1,1)$ is a local minimum.

(5) $\frac{\partial f}{\partial x}(x,y) = 3x^2 - 6x$, $\frac{\partial f}{\partial y}(x,y) = 6y^2 - 6y$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$ i.e. $3x^2 - 6x = 0 = 6y^2 - 6y$. This implies that $x = 0$ or $x = 2$ and $y = 0$ or $y = 1$. There are four critical points: $(0,0)$, $(0,1)$, $(2,0)$ and $(2,1)$. The matrices $D^2 f$ for them are

$$\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

This proves (Sylvester's criterion) that $(0,0)$ is a local maximum for f, $(0,1)$ and $(2,0)$ are saddles and $(2,1)$ a local minimum for f.

(6) $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 + 3x$. We are looking for the critical points of f so we want to solve the system of equations $\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$ i.e. $3x^2 + 3y = 0 = 3y^2 + 3x$ so $x^3 = -xy = y^3$ thus $x = y$. Therefore $0 = x^2 + y = x^2 + x$ so either $x = 0$ or $x = -1$.

In this case there are two critical points $(0,0)$ and $(-1,-1)$. The matrices $D^2f(x,y)$ for them are $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$. By Sylvester's criterion $(0,0)$ is a saddle for f and $(-1,-1)$ is a local maximum for f. □

11. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $F = (f_1, f_2)$, where $f_1(x,y)$ and $f_2(x,y)$ are selected from question 9 as follows: if your keys satisfy $k_1 \neq k_3$, then f_1 is the function in case (k_1) of question 9, and f_2 is the function in case (k_3) of question 9. If your keys satisfy $k_1 = k_3$ then f_2 is the function in case (k_4) of question 9 where $k_4 = (k_3 + 1) \bmod 6 + 1$.
Is your transformation F a C^1 function at $(2,1)$?
What is the value of $\det DF(2,1)$?
Is your transformation F a C^1 diffeomorphism on some ball with center $(2,1)$ and radius $\varepsilon > 0$?
Each question is worth 1 point.
We shall investigate one case because the arguments are the same in all of them and formally speaking there are many cases.
Let $F(x,y) = (3x^3 + 2y^2, x^5 - 2y^2)$. This map is a C^1 even C^∞ because its coordinates are C^∞. Then $DF(x,y) = \begin{pmatrix} 3x^2 & 4y \\ 5x^4 & -4y \end{pmatrix}$ hence $DF(2,1) = \begin{pmatrix} 12 & 4 \\ 80 & -4 \end{pmatrix}$ and therefore $\begin{vmatrix} 12 & 4 \\ 80 & -4 \end{vmatrix} = 12(-4) - 4 \cdot 80 = -368 \neq 0$. Since this determinant is different from 0 the map F is a local diffeomorphism on some ball centered at $(2,1)$ of sufficiently small radius (inverse function theorem). □

12. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function $F = (f_1, f_2)$ in question 11. Let $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function and let $h(x,y) = g \circ F(x,y)$. Given that $\nabla g(F(2,1)) = (1,1)$, calculate $\nabla h(2,1)$.

As we know from problem 11 the equality $DF(2,1) = \begin{pmatrix} 12 & 4 \\ 80 & -4 \end{pmatrix}$ holds. Also $F(2,1) = (26,30)$. By the chain rule

$DH(2,1) = Dg(26,30) \cdot DF(2,1) = (1,1) \cdot \begin{pmatrix} 12 & 4 \\ 80 & -4 \end{pmatrix} = (12 + 80, 4 - 4) = (92, 0)$. □