Quiz, April 7, 2020

Name and surname (type) ....

Student number (type)..

NOTE! The first step is to calculate your parameters a, b, r and s as follows. Let $p =$ **the third digit** of your student number and $q =$ **the last digit** of your student number, then

$$a = 1 + |p - q|, \qquad b = 2 + p + q, \qquad r = \frac{a}{b}, \qquad s = \frac{b}{a}.$$

Step 2. Answer the questions below using your values for the parameters a amd b:

1. Calculate the indefinite integral $\displaystyle\int \frac{1}{x^2(ax+b)}dx.$ answer: $\frac{a}{b^2}\ln\frac{|ax+b|}{|x|} - \frac{1}{bx} + C$

Solution. $\displaystyle\int \frac{1}{x^2(ax+b)}dx = \int \left(\frac{-a}{b^2x} + \frac{1}{bx^2} + \frac{a^2}{b^2(ax+b)}\right) dx = \frac{-a}{b^2}\ln|x| - \frac{1}{bx} + \frac{a}{b^2}\ln|ax+b| + C =$
$\displaystyle = \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| - \frac{1}{bx} + C.$ □

2. Calculate the definite integral $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos(x^2)dx.$ answer: $\frac{1}{2}$.

Solution. $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos(x^2)dx \xlongequal[du=2xdx]{u=x^2} \frac{1}{2}\int_0^{\pi/2} \cos u du = \tfrac{1}{2}\sin u\Big|_0^{\pi/2} = \tfrac{1}{2}\sin\tfrac{\pi}{2} - \tfrac{1}{2}\sin 0 = \tfrac{1}{2}.$ □

3. Calculate the definite integral $\displaystyle\int_0^{\infty} xe^{-ax}dx.$ answer: $\frac{1}{a^2}$

Solution. We shall integrate by parts. Le us start with indefinite integral.
$\int xe^{-ax}dx = -\frac{1}{a}e^{-ax}\cdot x + \frac{1}{a}\int e^{-ax}dx = -\frac{1}{a}e^{-ax}\cdot x - \frac{1}{a^2}e^{-ax} + C$. For evaluating the definite integral we may choose a number C as we want to. Let $C = 0$ and let $F(x) = -\frac{1}{a}e^{-ax}\cdot x - \frac{1}{a^2}e^{-ax}$. Recall that in all papers $a > 0$. We have $F(0) = -\frac{1}{a^2}$ and $\lim\limits_{x\to\infty} F(x) = 0$. The last equality is a consequence of the estimate $e^{ax} = \sum\limits_{n=0}^{\infty}\frac{(ax)^n}{n!} > \frac{1}{2}a^2x^2$ for $x > 0$ and therefore $xe^{-ax} = \frac{x}{e^{ax}} < \frac{2x}{a^2x^2} = \frac{2}{a^2x} \xrightarrow[x\to\infty]{} 0$.
One also may use the d'Hospital's rule instead of the estimate. This implies that

$$\int_0^{\infty} xe^{-ax}dx = \lim_{x\to\infty} F(x) - F(0) = 0 + \frac{1}{a^2} = \frac{1}{a^2}.$$

4. Is the set $\{(x,y)\in\mathbb{R}^2:\quad x^2+y^2\leqslant s,\ ax+by<b\}$:

closed	**Yes/No:**	No
open	**Yes/No:**	No
bounded	**Yes/No:**	Yes
compact	**Yes/No:**	No
connected	**Yes/No:**	Yes
convex	**Yes/No:**	Yes

Solution. Let $A=\{(x,y)\in\mathbb{R}^2:\quad x^2+y^2\leqslant s,\ ax+by<b\}$. A is not a closed set because $(0,\frac{n-1}{n})\in A$ for $n=1,2,3,\ldots$ while $\lim\limits_{n\to\infty}(0,\frac{n-1}{n})=(0,1)\notin A$. A is not open because $(-\sqrt{s},0)\in A$ and no point on x–axis to the left of $(-\sqrt{s},0)$ is in A – the first inequality in the definition of A is n ot satisfied. A is bounded since it is contained in the disk of radius $\sqrt{s}$ centered at $(0,0)$. A is not compact since it is not closed. It follows from the definition of convexity that A is a convex set: each segment with ends in A is entirely contained in A. A is connected because each convex set is connected.

Uwaga 0.1 In general every set which is convex is also connected. It is not hard to prove that if every 2 points of a set B can be joined with a path contained in B then the set is connected. In the almost simplest case you may think that a path is a sequence of straight line segments S_1, S_2,. . . ,S_k such that the end of S_i is the begining of S_{i+1} for $i=1,2,3,...,k-1$. Sometimes such path is called a polygonal chain, see https://en.wikipedia.org/wiki/Polygonal_chain

5. Find the following limit or state that it does not exist

$$\lim_{n\to\infty}\left(\sqrt[n]{n^2},\ \frac{\ln n}{\sqrt{n}},\ \left(1+\frac{b}{n}\right)^n\right).\qquad\text{answer:}\qquad (1,0,e^b)$$

Solution. $\lim\limits_{n\to\infty}\sqrt[n]{n^2}=\left(\lim\limits_{n\to\infty}\sqrt[n]{n}\right)^2=1^2=1$. $\frac{\ln n}{\sqrt{n}}=\frac{4\ln\sqrt[4]{n}}{\sqrt{n}}<\frac{4\sqrt[4]{n}}{\sqrt{n}}=\frac{4}{\sqrt[4]{n}}\xrightarrow[n\to\infty]{}0$ thus $\lim\limits_{n\to\infty}\frac{\ln n}{\sqrt{n}}=0$. $\lim\limits_{n\to\infty}\left(1+\frac{b}{n}\right)^n=e^b$. □

6. Find the following limit or state that it does not exist

$$\lim_{(x,y)\to(0,0)}\frac{xy^2}{ax^4+by^2}\qquad\text{answer:}\qquad 0.$$

Solution. We have $\left|\frac{xy^2}{ax^4+by^2}\right|=|x|\cdot\frac{y^2}{ax^4+by^2}=\frac{|x|}{b}\cdot\frac{by^2}{ax^4+by^2}\leqslant\frac{|x|}{b}\xrightarrow[(x,y)\to(0,0)]{}0$.

7. Let $A = (0,0)$, $B = (2,6)$ and $C = (5,0)$. Is the angle ABC smaller than $90°$?

Yes/No: Yes

Solution. The dot product of the vectors $A - B = (-2, -6)$ and $C - B = (3, -6)$ equals $(-2)\cdot 3 + (-6)\cdot(-6) = 30 > 0$ so the cosine of the angle made by these vectors is positive. Thus the angle is less than $90°$.

Solution 2. Let $D = (2,0)$. Then $\tan\sphericalangle ABD = \frac{1}{3}$ and $\tan\sphericalangle CBD = \frac{1}{2}$. This implies that $\tan\sphericalangle ABC = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1$. This implies that $\sphericalangle ABC = 45° < 90°$. □

8. Let $A = (0,0)$, $B = (2,6)$ and $C = (5,0)$. Let $X = (x_1, y_1)$, $Y = (x_2, y_2)$, $Z = (x_3, y_3)$ be points that lie on the straight line segments AB, BC and CA respectively. Let $K =\subset \mathbb{R}^6$ be the set consisting of the sequences $(x_1, y_1, x_2, y_2, x_3, y_3)$. Let $f(x_1, y_1, x_2, y_2, x_3, y_3) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} + \sqrt{(x_2-x_3)^2 + (y_2-y_3)^2} + \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2} = \|X - Y\|_2 + \|Y - Z\|_2 + \|Z - X\|_2$ for $(x_1, y_1, x_2, y_2, x_3, y_3) \in K$.

Does the function $f\colon K \longrightarrow \mathbb{R}$ attains its least upper bound? **Yes/No:** ....

Does the function $f\colon K \longrightarrow \mathbb{R}$ attains its greatest lower bound? **Yes/No:** ....

Solution. We have $0 \leqslant x_1 \leqslant 2$, $0 \leqslant y_1 \leqslant 6$, $2 \leqslant x_2 \leqslant 5$, $0 \leqslant y_2 \leqslant 6$, $0 \leqslant x_3 \leqslant 5$, $y_3 = 0$. This proves the set K is bounded. It is also closed. This follows from the fact that the straight line segment which contains its end points is closed. The function in question is continuous: if (X, Y, Z) and (X', Y', Z') are two triples then $\|X-Y\|_2 + \|Y-Z\|_2 + \|Z-X\|_2 - (\|X'-Y'\|_2 + \|Y'-Z'\|_2 + \|Z'-X'\|_2) \leqslant$ $\leqslant \|X-X'\|_2 + \|Y-Y'\|_2 + \|Y-Y'\|_2 + \|Z-Z'\|_2 + \|Z-Z'\|_2 + \|X-X'\|_2 =$ $= 2\left(\|X-X'\|_2 + \|Y-Y'\|_2 + \|Z-Z'\|_2\right)$. This inequality proves the continuity of the function f. A continuous function defined on a compact set attains its sup and inf. This Weierstrass maximum/ minimum theorem.

9. Let $f(x,y) = x^2(1+y)^3 + y^2$.

Find all critical points of $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$. answer: $(0,0)$

Find all points at which $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ has a local minimum. answer: $(0,0)$

Find $\inf\{f(x,y)\colon \quad (x,y) \in \mathbb{R}^2\}$. answer: $-\infty$

Find $\inf\{f(x,y)\colon \quad (x,y) \in \mathbb{R}^2,\ y > -1\}$. answer: 0

Find $\sup\{f(x,y)\colon \quad (x,y) \in \mathbb{R}^2,\ y > -1\}$. answer: ∞

Solution. $\frac{\partial f}{\partial x} = 2x(1+y)^3$, $\frac{\partial f}{\partial y} = 3x^2(1+y)^2 + 2y$. If $\frac{\partial f}{\partial x} = 2x(1+y)^3 = 0$ then either $x = 0$ or $y = -1$. If also $\frac{\partial f}{\partial y} = 3x^2(1+y)^2 + 2y = 0$ then $y = 0$ in both cases but it is impossible in the second case. So f has one critical point namely $(0,0)$. If $y > -1$ then $f(x,y) > 0$ with one exception: $f(0,0) = 0$. This prove that at $(0,0)$ the function f has a local minimum and if f is restricted to the half–plane $y > -1$ then $0 = f(0,0)$ is its smallest value. $f(1,y) = (1+y)^3 + y^2$ so it is a cubic polynomial in y so it is unbounded from below and from above: $\lim\limits_{y\to\infty} (1+y)^3 + y^2 = +\infty$ and $\lim\limits_{y\to-\infty} (1+y)^3 + y^2 = -\infty$.

This justifies answers to the third and to the fifth questions. $\square$

10. Let $f(x,y) = \cos x \cdot \tan y$.

Does f have a local maximum at the point $(0,0)$? **Yes/No:** No

Does f have a local minimum at the point $(0,0)$? **Yes/No:** No

f neither has local minimum nor local maximum at $(0,0)$. **Yes/No:** Yes

Solution. $\frac{\partial f}{\partial y} = \cos x \cdot (1+\tan^2 y)$, so $\frac{\partial f}{\partial y}(0,0) = 1 \neq 0$. This proves that $(0,0)$ is not a critical point of f. Therefore f has neither local minimum nor local maximum at $(0,0)$. $\square$

11. Let $f(x,y) = \sin^2 x + 2a\ln(1+x)\tan y - 2\cos y$.

For what $a \in \mathbb{R}$ the equality grad $f(0,0) = (0,0)$ holds? answer: $a \in \mathbb{R}$

For what $a \in \mathbb{R}$ the function f has a local minimum at the point $(0,0)$? answer: $-1 < a < 1$

For what $a \in \mathbb{R}$ the function f has a local maximum at the point $(0,0)$? answer: $a \in \emptyset$

For what $a \in \mathbb{R}$ the function f has a saddle at the point $(0,0)$? answer: $a \notin (-1,1)$

Solution. $\frac{\partial f}{\partial x} = 2\sin x\cos x + \frac{2a\tan y}{1+x}$, $\frac{\partial f}{\partial y} = 2a\ln(1+x)(1+\tan^2 y) + 2\sin y$. From these equalities it follows that grad $f(0,0) = (0,0)$ for all $a \in \mathbb{R}$.

$\frac{\partial^2 f}{\partial x^2} = 2\cos^2 x - 2\sin^2 x - \frac{2a\tan y}{(1+x)^2}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{2a}{1+x}(1+\tan^2 y)$, $\frac{\partial^2 f}{\partial y} = 4a\ln(1+x)\tan y(1+\tan^2 y) + 2\cos y$.

From these equalities it follows that $D^2 f(0,0) = \begin{pmatrix} 2 & 2a \\ 2a & 2 \end{pmatrix}$. The determinant of this matrix equals $4 - 4a^2 = 4(1-a^2)$ so the determinant is positive iff $a^2 < 1$. Since the entry at the left upper corner is positive too the matrix is positively defined for $-1 < a < 1$ which proves that f has a local minimum for such a. If $a^2 > 1$ then the determinant is negative so the function has a saddle at $(0,0)$.

Now let us look at $a = 1$. Then $f(x,y) = \sin^2 x + 2\ln(1+x)\tan y - 2\cos y$. Now we have $f(x,-x) = \sin^2 x - 2\ln(1+x)\tan x - 2\cos x$. We have $\frac{d}{dx}\big(\sin^2 x - 2\ln(1+x)\tan x - 2\cos x\big) =$ $=2\sin x\cos x - \frac{2\tan x}{1+x} - 2\ln(1+x)(1+\tan^2 x) + 2\sin x$. Then $\frac{d^2}{dx^2}\big(\sin^2 x - 2\ln(1+x)\tan x - 2\cos x\big) =$ $=2\cos^2 x - 2\sin^2 x + \frac{2\tan x}{(1+x)^2} - \frac{2(1+\tan^2)x}{1+x} - \frac{2(1+\tan^2 x)}{1+x} + 4\ln(1+x)\tan x(1+\tan^2 x) + 2\cos x =$ $= 2\cos^2 x - 2\sin^2 x + \frac{2\tan x}{(1+x)^2} - \frac{4(1+\tan^2 x)}{1+x} + 4\ln(1+x)\tan x(1+\tan^2 x) + 2\cos x$. Substitute 0 for x in this formula. The result is 0. So the first and the second derivatives of $x \mapsto f(x,-x)$ vanish. Let us compute the third derivative of this function at 0 only. Obviously $\frac{d}{dx}(2\cos^2 x - 2\sin^2 x + 2\cos x)$ is 0 at 0. It is so because the function attains its maximal value at 0. The derivative of $\frac{2\tan x}{(1+x)^2} + 4\ln(1+x)\tan x(1+\tan^2 x)$ at 0 is 2 because $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$ – to obtain this result we use the definition of the derivative. The last part is $-\frac{4(1+\tan^2 x)}{1+x} = \frac{-4}{1+x} - \frac{-4\tan^2 x}{1+x}$. The derivative of $\frac{-4}{1+x}$ equals $\frac{4}{(1+x)^2}$ so at 0 it is 4. $\lim\limits_{x\to 0}\frac{-4\tan^2 x}{x(1+x)} = \lim\limits_{x\to 0}\frac{-4\tan x}{x} \cdot \lim\limits_{x\to 0}\frac{\tan x}{(1+x)} = -4 \cdot 0 = 0$ so by definition of the derivative we know that the derivative at 0 of $\frac{-4\tan^2 x}{1+x}$ is 0. Therefore the third derivative of $x \mapsto f(x,-x)$ at 0 is 6. Thus proves that the function assumes positive and negative values at any neighbourhood of 0. So the function f has a saddle at $(0,0)$.

The same method applies to $f(x,y) = \sin^2 x - 2\ln(1+x)\tan y - 2\cos y$. The only difference is that this time we look at $f(x,x)$ but this the same function we just finished to investigate. $\square$

Uwaga 0.2 Instead of computing the third derivative one might use Taylor expansions. This would give the same result faster. $\sin x = x - \frac{x^3}{6} + o(x^3)$. This is an abbreviation of the sentence $\lim\limits_{x\to 0} \frac{\sin x - (x - \frac{x^3}{6})}{x^3} = 0$. From this it follows that $\sin^2 x = (x - \frac{x^3}{6} + o(x^3))^2 = x^2 + o(x^3)$. Then $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$ and $\tan x = x + \frac{x^3}{3} + o(x^3)$. Therefore $\ln(1+x)\tan x =$ $=(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3))(x + \frac{x^3}{3} + o(x^3)) = x^2 - \frac{x^3}{2} + o(x^3)$. The last expansion is $\cos x = 1 - \frac{x^2}{2} + o(x^3)$. The final result is

$\sin^2 x - 2\ln(1+x)\tan x - 2\cos x = x^2 + o(x^3) - 2(x^2 - \frac{x^3}{2} + o(x^3)) - 2(1 - \frac{x^2}{2} + o(x^3)) = -2 + x^3 + o(x^3)$. The function $-2 + x^3$ assumes at any neighbourhood of 0 values less than -2 (for negative x) and values greater than -2 (for positive x). The remainder $o(x^3)$ is too little for small x to be able to change the sign of x^3. The nonexistence of local extremum follows. $\square$