A new 3-color criterion for planar graphs

Krzysztof Diks, Lukasz Kowalik, Maciej Kurowski
{diks, kowalik, kuros}@mimuw.edu.pl
Institute of Informatics, Warsaw University
Banacha 2, 02-097 Warsaw
Poland

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Abstract

We present a new general 3-color criterion for planar graphs. Using this criterion we characterize a broad class of 3-colorable planar graphs and provide a corresponding linear time 3-coloring algorithm. We also fully characterize infinite 3-colorable planar graphs.

1 Introduction

The problem of coloring the vertices of a graph using few colors has given rise to one of the most intensively studied areas of graph theory. A frequently encountered special case is that in which the graph to be colored is planar. Computing a coloring that uses the smallest possible number of colors is known to be an NP-complete problem, even when restricted to the class of planar graphs and 3 colors. More precisely it is an NP-complete problem to decide in general whether a given planar graph is 3-colorable [GJS]. On the other hand there is known that every planar graph is 4-colorable [AH1, AH2], this is the famous ,,4-color theorem". Hence it is natural to characterize those planar graphs which are 3-colorable. The first 3-color criterion was formulated by Heawood in 1898 and it is known as the Three Color Theorem [Hea, Ste]: *A maximal planar graph is vertex colorable in three colors if and only if all its vertices have even degrees.* Obviously this theorem implies a very simple algorithm for checking 3-colorability of maximal planar graphs. As the problem of 3-colorability of planar graphs is NP-complete one cannot rather expect any ,,effective" 3-color criterion for general planar graphs. On the other hand there is a ,,simple", general 3-color criterion which does not lead to an efficient 3-color algorithms: *A planar graph is vertex colorable in three colors if and only if it is a subgraph of a maximal planar graph in which all vertices have even degrees.* This theorem was already known to Heawood and discovered independently by several authors – see the comprehensive survey written by Steinberg [Ste].

In this paper we introduce a new general 3-color criterion which can be efficiently checked for a broad class of planar graphs. Our criterion generalizes the Heawood's Three Color Theorem. In order to get this result we define a new class of planar graph colorings, so called edge-side colorings, and prove that this new type of coloring is equivalent to the ordinary vertex 3-coloring. We present linear algorithms for converting edge-side colorings into 3-colorings and vice versa. Using our criterion we are able to characterize 3-colorable triangulations with holes, i.e. plane graphs in which each vertex touches at most one non-triangular face. We provide a linear time algorithm for 3-coloring such graphs. Our criterion allows also to formulate sufficient and necessary conditions for 3-coloring infinite planar graphs.

The rest of the paper is organized as follows. In section 2 we introduce basic definitions and notation. In section 3 we formulate and prove our new 3-color criterion and in section 4 we show its applications. Section 5 contains the algorithms for converting 3-colorings into edge-side colorings and vice versa.

2 Basic definitions and notation

It is known that a graph is 3-colorable iff all its biconnected components are 3-colorable. In the sequel, if it is not stated explicitly, saying a graph we mean a biconnected finite multigraph of at least three vertices but without selfloops.

A *plane graph* is a graph whose vertex set is a point set in the plane and the edges are Jordan curves such that two different edges have, at most, end points in common. A graph is called *planar* if it can be embedded in the plane, i.e. if it is isomorphic to a plane graph.

Let C be a simple cycle in a plane graph G. The cycle C divides the plane into two disjoint, open domains – the *interior C-domain* (homeomorphic to a disc) and the *exterior C-domain*. The set consisting of all vertices of G belonging to the interior C-domain and of all edges crossing this domain will be denoted by $\operatorname{Int} C$. If v is a vertex in C then the number of the graph neighbors of v lying in the interior C-domain is called the *internal degree* of v with respect to C and it is denoted by $dIn(C,v) = |\{(v,w) \in E(G) \ : \ w \in \operatorname{Int} C\}|$. We define the internal degree of the cycle C as the sum of the internal degrees of all its vertices. This sum is denoted by $dIn(C) = \sum_{v \in C} dIn(C,v)$.

A *face* in a plane graph G is a C-domain (interior or exterior), for some cycle C, without any vertices and edges inside. Only one face is unbounded and it is called the *outer* face. Similarly, its boundary cycle is called the *outer* one.

A *triangulation* is a plane graph in which the boundary cycle of every face is a triangle (3-cycle). A biconnected plane graph in which all the boundary cycles, except at most one, are triangles is called a *near-triangulation*. W.l.o.g. we will consider this boundary cycle to be the outer one.

A graph is *even* if all its vertices have even degrees. A near-triangulation is *internally even* if all its vertices different from those on the outer cycle have even degrees.

3 A new 3-color criterion

In 1898 Heawood [Hea] proved a theorem characterizing (finite) 3-colorable triangulations:

Theorem 1 (Three Color Theorem). *A (finite) triangulation is 3-colorable if and only if it is even.*

This criterion applies only to the *maximal* planar graphs which are isomorphic to triangulations. It allows to check in a very simple manner whether a given maximal planar graph is 3-colorable. One can ask a natural question: can it be generalized to all planar graphs? Unfortunately, since checking 3-colorability is an NP-complete problem even in planar case, we cannot rather expect any polynomial criterion for general planar graphs. But there are general criteria which allow checking 3-coloring of planar graphs in some special cases. As stated in [Ste] such a general criterion was already known to Heawood [Hea], nevertheless it was not widely known and has been independently discovered and proved several times, e.g. in [Kr1], [Kr2], [Mar]. The criterion follows:

Theorem 2 (Heawood's 3-color criterion). *Let G be a plane graph. The following two conditions are equivalent:*

(i) G is 3-colorable.

(ii) There exists an even triangulation H such that G is a subgraph of H, i.e. $H \supseteq G$.

Moreover, every 3-coloring of a plane graph G can be extended to a 3-coloring of some even triangulation $H \supseteq G$.

As we can see the 3-color criterion stated above tells us nothing about the structure of the graph under consideration. In this section we provide a new type of graph coloring, called edge-side coloring, which is equivalent to the vertex 3-coloring but additionally reflects some structural properties of the graph. This new feature will allow us to characterize a new, broad class of 3-colorable planar graphs which are recognizable and 3-colorable in linear time.

We start from a few indispensable definitions.

Let G be a plane graph, f a face in G and e an edge on the boundary cycle of f. The pair $s = (e,f)$ is called a *side of edge e in face f* (or shortly a *side*). We say also that side s touches face f. If vertex v is an end point of e then side s is said to be incident with v. Observe that every edge in a biconnected graph has exactly two sides.

Let G be a plane graph and S be the set of all sides in G. *Edge-side coloring* of G is an arbitrary function

$$m : S \longrightarrow \{black, white\}$$

Edges with one side black and the other side white are called *b-w edges*. The other edges are called *one color edges* and can be of type *b-b* or *w-w* depending on the colors of their sides, black or white, respectively.

We call an edge-side coloring of a plane graph G *proper* if the following two conditions are satisfied:

(i) for each face f in G the numbers of white and black sides touching f are congruent (equal) mod 3;

(ii) each vertex v in G is incident with an even number of one color edges.

We say that a plane graph G is *edge-side colorable* if its edge-sides can be properly colored.

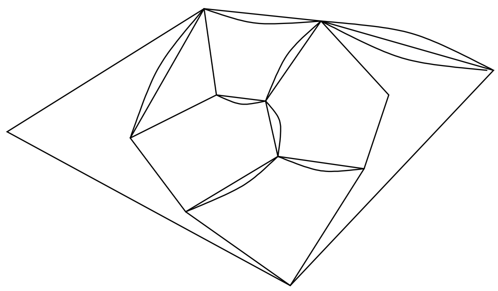


Figure 1: A proper edge-side coloring – arcs indicate black sides.

Now we can state the main theorem of the paper.

Theorem 3 (3-color criteria). *Let G be a (biconnected) plane graph. The following three conditions are equivalent:*

(i) G is 3-colorable.

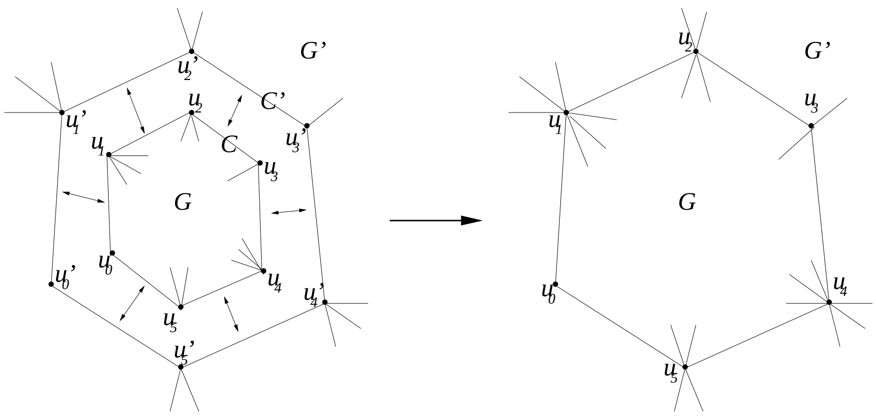
(ii) There exists an even triangulation $H \supseteq G$.

(iii) G is edge-side colorable.

The equivalence of conditions (i) and (ii) was proved by Heawood (see Theorem 2). The proof of the equivalence of (ii) and (iii) is our main contribution to this paper. We start from a few observations on internally even near-triangulations.

Lemma 1. *Every internally even near-triangulation is 3-colorable.*

Proof. Let C be the outer cycle of a near-triangulation G. Let us take a separate embedding G' of G in which the cycle C' corresponding to C is not longer the outer one. Now we build a new plane graph H from G' placing the entire graph G in the interior C'-domain and identifying the corresponding vertices and edges of both cycles C and C'. See the figure below.



One can easily check that the graph H is an even plane triangulation and hence it is 3-colorable by the Heawood's Three Color Theorem. Since $G \subseteq H$ it is also 3-colorable. $\square$

Lemma 2. *Let G be an even near-triangulation with the outer cycle C. Then $|C| \equiv 0 \pmod 3$.*

Proof. By Lemma 1, G is 3-colorable. Let $C = v_0 v_1 \dots v_{|C|-1} v_0$ and let $\mathcal{K}$ be an arbitrary 3-coloring of G. We will show that one can rename the colors in $\mathcal{K}$ in such a way that $\mathcal{K}(v_i) = (i \bmod 3) + 1$, for every $i = 0 \dots |C| - 1$.

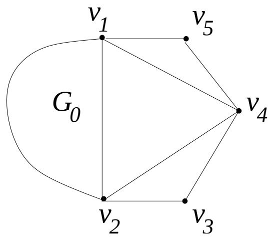
Let v be an arbitrary vertex in C and let x and y be its neighbors such that x, y and v are incident to the same internal (not infinite) triangular face in G. Then, x, y, and v have different colors. Now one can observe that every two successive neighbors, in a sequence of all neighbors of v listed in clockwise order, have different colors. Since the degree of v is even its neighbors on C have different colors. As a result we get $\mathcal{K}(v_i) = (i \bmod 3) + 1$ (possibly, after renaming the colors) which implies $|C| \equiv 0 \pmod 3$. $\square$

Lemma 3. *For every $i \geq 3$ such that $i \equiv 0 \pmod 3$ there exists a finite even near-triangulation with the outer cycle of length i.*

Proof. The proof is by induction on i. For $i = 3$ it suffices to take K_3.

Induction step: by the induction hypothesis, there exists G_0 – a finite, even near-triangulation with the outer cycle of length $i - 3$. Let v_1, v_2 be arbitrary adjacent vertices in the outer cycle of G_0. Then

$$G = G_0 \cup \{v_3, v_4, v_5\} \cup \{v_2 - v_3, v_3 - v_4, v_4 - v_5, v_5 - v_1, v_2 - v_4, v_1 - v_4\}$$



is an even near-triangulation and its outer cycle has length i. $\square$

Let C be a simple cycle and let m_C be an arbitrary edge 2-coloring of C,

$$m_C : E(C) \longrightarrow \{black, white\}$$

We say that coloring m_C is *good* if $|m_C^{-1}(black)| \equiv |m_C^{-1}(white)| \pmod 3$.

Let G be a graph and let C be a simple cycle in G. We say that a good coloring m_C of C *corresponds* with G if the following holds: for every vertex v in C the edges in $E(C)$, incident with v, have different colors if and only if degree $d_G(v)$ of v in G is odd.

Lemma 4 (Key lemma).

(i) For every internally even near-triangulation G with the outer cycle C there exists a good 2-coloring m_C of C corresponding with G.

(ii) For every good 2-coloring m_C of C there exists an internally even near-triangulation G with the outer cycle C and such that m_C corresponds with G.

Proof of (i). Since G is internally even, the number of vertices of C with odd degrees is even. Let

$$v_1, v_2, \dots, v_{2k-1}, v_{2k}$$

be a list of all such vertices in the clockwise order on C. For each $i = 1, 2, \dots, k$, color all edges on C between (v_{2i-1}, v_{2i}) black. Color the remaining edges white. (See Fig. 2) Observe that each vertex in C is adjacent with edges of different colors if and only if it has odd degree.

Let b be the number of black edges on C and let w be the number of white edges on C. After extending G by triangles formed in the external (infinite) face and with the black edges

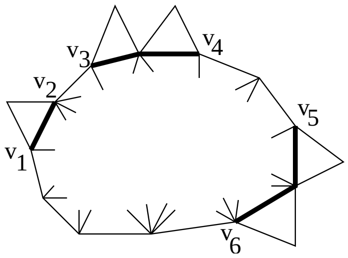


Figure 2: Constructing a good 2-coloring

as the triangle bases (as shown in Fig. 2) we get an even near-triangulation. By Lemma 2 the outer cycle of this triangulation has length $\equiv 0 \pmod 3$. Hence $2b + w \equiv 0 \pmod 3$ and finally $b \equiv w \pmod 3$. □

Proof of (ii). Denote the number of black and white edges of C by b and w, respectively. We form a triangle on each black edge e in the interior C-domain as shown in Fig. 3. As the result we get a graph H. Observe that vertex v of H has odd degree if and only if it is adjacent in C to the edges of different colors.

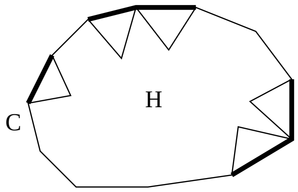


Figure 3: The graph H with the outer cycle C – constructing a near-triangulation

Let f be the only face of H different from the added triangles and placed in the interior C-domain. The length of the f's boundary cycle is $2b + w \equiv 0 \pmod 3$. By Lemma 3, one can triangulate this face and obtain a required internally even near-triangulation G. □

Lemma 5. *Let G be a finite, biconnected plane graph. Let H be an even triangulation (possibly infinite but locally finite) such that $G \subseteq H$. If for every face f of G with facial cycle C_f there is a good edge 2-coloring corresponding with $\mathrm{Int}_H\, C_f \cup C_f$, the graph G is edge-side colorable.*

Proof. For every face f of G with facial cycle C_f let m_{C_f} be a a good edge 2-coloring corresponding with $\mathrm{Int}_H\, C_f \cup C_f$. Now we can construct an edge-side coloring assigning to each side (f, e) the color $m_{C_f}(e)$. Obviously this coloring satisfies the first condition in the definition of proper edge-side coloring. We shall show the latter one.

Let v be an arbitrary vertex in G. Consider a facial cycle C' of a face f' such that $v \in V(C')$ and $dIn_H(C', v)$ is even. Then if $e_1, e_2 \in E(C')$ are incident to v, the sides (e_1, f') and (e_2, f') have the same color. On the other hand, when $dIn_H(C', v)$ is odd, the sides have different colors and there is odd number of such facial cycles containing v (since $d_H(v)$ is even). Denote the number of black sides incident with v by $B(v)$. Now we see that if $d_G(v)$ is odd, $B(v)$ is odd. On the other hand, if $d_G(v)$ is even, $B(v)$ is even. If follows that $d_G(v) + B(v)$ is even, which is equivalent to the statement that the number of one color edges incident with v is even.

We have just showed that the second condition in the definition of proper edge-side coloring is satisfied, which completes the proof. □

Now we are ready to prove the part (ii)↔(iii) of our main theorem.

Proof.
(ii)$\longrightarrow$(iii)
Assume that there exists an even triangulation $H \supseteq G$. Observe that since G is 2-connected, every face is bounded by a simple cycle. For each face f with facial cycle C_f we can apply lemma 4 to get a good edge 2-coloring m_C of E(C) corresponding with the near-triangulation $C_f \cup \mathrm{Int}_H\, C_f$. Now we can apply lemma 5 to obtain a proper edge-side coloring of G, what completes the proof of (ii)$\longrightarrow$(iii).

(ii)$\longleftarrow$(iii)
Assume that G is properly edge-side colored. By Lemma 4 one can triangulate (i.e. divide into triangles) each face into an internally even near-triangulation getting a triangulation $H \supseteq G$. Let v be an arbitrary vertex of G. Denote by $F(v)$ the number of faces incident with v for which the odd number of edges ending in v was added during the the triangulation process. Similarly as in the proof of the reverse implication one can show that $d_G(v) + F(v)$ is even what follows from the fact that for every $v \in V(G)$, $d_G(v) + B(v)$ is even. Finally it implies that for every vertex v, $d_H(v)$ is even, what means that H is an even triangulation. $\square$

As the result we have got a new 3-color criterion for general planar graphs. In fact, using this criterion for checking whether an arbitrary plane graph is 3-colorable is equally hard as trying to find a proper 3-coloring of a given graph. However in the next section we describe some applications of our theorem showing a few classes of planar graphs for which the new criterion can be effectively checked.

4 Applications

One can expect that the criterion formulated in section 3 can be effectively checked for graphs that are "highly triangulated", i. e. when *a lot* of faces are triangles. Moreover, if such a graph has a special structure it can be colored using a greedy algorithm. We define formally a class of graphs for which the greedy algorithm works well. We call a plane graph G *triangle connected* if each vertex of G is incident with a triangular face and the subgraph of the graph dual to G induced by the triangular faces is connected.

In the following subsections we present the greedy algorithm and three classes of graphs for which effective 3-color criteria can be formulated. Our general criterion can be also used to show that plane graphs with face lengths of multiple of three are 3-colorable.

4.1 The greedy algorithm

Given a planar, triangle connected graph (without its planar embedding) the algorithm computes its 3-coloring or reports that it doesn't exist. The algorithm runs in linear time. For each vertex v the set PossibleColors(v) contains colors which are still admissible for v; S represents the set of vertices for which set PossibleColor contains at most one color. Algorithm uses operation `RESTRICT(v)` which restricts the set of admissible colors for neighbors of v.

OPERATION `RESTRICT`(v)::
for each u **in** Neighbors(v) **do**
  if Col(u) = -1 **then**
  begin
    PossibleColors(u).Remove(Col(v))
    if |PossibleColors(u)| ≤ 1 **then** S.Add(u)
  end

ALGORITHM `GREEDY`::
for each v **in** $V(G)$ **do**
begin
  PossibleColors(v) := {1, 2, 3};
  Col(v) := -1 {undefined}
end
S := $\emptyset$

```
(p, q) := arbitrary edge of G
Col(p) := 1
Col(q) := 2
RESTRICT(p)
RESTRICT(q)

while not S.Empty do
begin
  v := S.Remove
  if |PossibleColors(v)| ≠ 1 then
    Exit(G is not 3-colorable)
  else
  begin
    Col(v) := PossibleColors(v).Get
    RESTRICT(v)
end while
```

4.2 Triangulations with holes

A biconnected plane graph is called a *triangulation with holes* if every of its vertices is incident wit at most one non-triangular face, i.e. face of length at least 4.

Remark. Every triangulation with holes is triangle connected.

Proof. Let G be a triangulation with holes. Consider two triangular faces f, h sharing a common vertex v. Vertex v is incident with at most one non-triangular face and subsequently f and h can be connected by a sequence formed of triangular faces, where each two successive faces share a common edge.

Now let f and h be two arbitrary triangular faces of G. Since G is connected, f and h can be connected by path $e_1, e_2, \ldots, e_k$ where e_i are edges of G. Each edge belongs to at least one triangular face, denote such face for e_i by t_i. Every two successive faces t_i, t_{i+1} share a vertex, and additionally t_1 shares a vertex with f and t_k shares a vertex with h. Hence we conclude that f and h are connected by a path formed of triangular faces, where each two successive faces share a common edge. □

Triangulations with holes have the following interesting property. Let G a triangulation with holes and let f be a face in G. Let C_f be the facial cycle of f. Then for every vertex v in C_f and an arbitrary even triangulation $H \supseteq G$ the parity of $dIn_H(C_f, v)$ is the same. It follows that there is exactly one edge-side coloring for every triangulation with holes (if not to consider isomorphic ones). This implies a very simple 3-color criterion for triangulations with holes.

We say that a triangulation with holes is internally even when the degree of every vertex incident with triangular faces only is even.

Theorem 4. *A triangulation with holes G is 3-colorable if and only if*

(i) it is internally even,

(ii) for every face f with a facial cycle C_f there exists a good edge 2-coloring of C_f corresponding with G.

Proof. The proof follows immediately from Theorem 3. It suffices to observe that if f is a face of length 3, one can color all its sides (e, f) black, but for faces f with facial cycles C_f of lengths $\neq 3$ every side (e, f) should be colored with color $m_{C_f}(e)$. □

4.3 Near-triangulations and outerplanar graphs

Obviously near-triangulations are triangulations with holes. Informally speaking a near-triangulation is a *triangulation with only one hole*. It gives a very simple 3-color criterion for near-triangulations:

Theorem 5. *A near-triangulation is 3-colorable if and only if it is internally even.*

Proof. Lemma 4 implies that the second condition of the Theorem 4 is always satisfied for near-triangulations. □

As every outerplanar graph is a subgraph of a certain internally even near-triangulation we immediately get the following known result:

Corollary 1. *Outerplanar graphs are 3-colorable.*

4.4 Planar graphs with face lengths of multiple of three

The following theorem ([Ore], [Ste]) easily follows from our criterion:

Theorem 6. *Let G be a graph embedded in the plane in such a way that the number of edges in the boundary of each face is a multiple of 3. If G is even then G is 3-colorable.*

Proof. It suffices to color all edge-sides in G black and to apply Theorem 3. □

4.5 Infinite triangulations

It is surprising that we can apply our criterion to infinite plane graphs. An *infinite triangulation* is an infinite plane graph with all faces being triangles. We will consider only locally finite triangulation where degrees of all vertices are finite. An *edge accumulation point* (shortly EAP) of an infinite plane graph G is a point P such that for every positive real number ϵ there are infinitely many edges of G with Euclidean distance from P less than ϵ. We will show that the Three Color Theorem holds also for EAP-free infinite triangulations.

Theorem 7. *An EAP-free infinite triangulation is 3-colorable if and only it is even.*

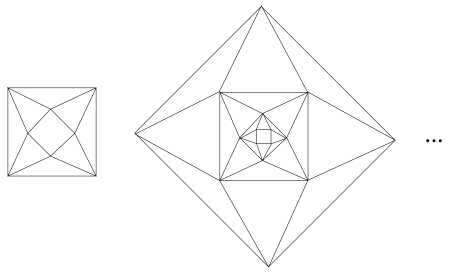
Proof. Assume that EAP-free infinite triangulation G is 3-colorable. Let v be a vertex of odd degree. Since arbitrary successive neighbors (in clockwise order) of v are adjacent, they have different colors. As there is an odd number of neighbors of v, we need 3 colors to color them and there is no color left for v. We have just proved implication ($\longrightarrow$).

Assume that G is even. Let v_0 be an arbitrary vertex of G. We will define a sequence of graphs

$$G_0 \subset G_1 \subset G_2 \subset G_3 \ldots$$

Let $V(G_0) = \{v_0\}$ and $E(G_0) = \emptyset$. Let W_i be the set of vertices with the graph distance at most i from v. Since graph $G(W_i)$ is finite, it has an infinite face f_i with the facial cycle C_i. Obviously there are not cut vertices in $G(W_i)$. Therefore C_i is a simple cycle. We define G_i as $C_i \cup \mathrm{Int}_G\, C_i$. Since G is EAP-free for every natural i, G_i is a finite graph. Moreover G_i is an internally even near-triangulation. It follows from Theorem 5 that graphs G_i are 3-colorable. Since $G_{i-1} \subseteq G_i$, for $i > 1$, and 3-colorings of G_i and G_{i-1} are unique (i. e. defining the unique partition of vertices into 3 independent subsets), we can construct 3-colorings $\mathcal{K}_i$ for graphs G_i, $i = 0, 1, 2, \ldots$, such that $\mathcal{K}_{i|G_{i-1}} = \mathcal{K}_{i-1}$. Now we can define a 3-coloring of G as $\mathcal{K}(u) = \mathcal{K}_{d(v_0,u)}(u)$. □

It is easy to show examples of infinite even triangulations with EAP that are even, but *not* 3-colorable:



In the sequel we use the following well-known fact.

Fact. Let G be an infinite graph. If every finite subgraph of G is k-colorable then G is k-colorable.

Theorem 8. *Let G be an infinite but locally finite triangulation (not necessarily EAP-free). G is 3-colorable if and only if*

(i) G is even,

(ii) for every simple cycle C in G there exists a good edge 2-coloring of C corresponding with $C \cup \mathrm{Int}_G C$.

Proof. Assume that G is 3-colorable then obviously G must be even. Now let us consider an arbitrary cycle C in G. Let $V_H \subset V$ be a set of vertices defined as follows: $v \in V_H$ if and only if v has a neighbor in $V(C)$ and $v \in \mathrm{Int}_G C$. Let $H = G(V_H)$. As $H \subset G$, H is 3-colorable. It is easy to see that H is a biconnected graph. Therefore we can apply Theorem 3 and get a required edge 2-coloring m_C corresponding with $C \cup \mathrm{Int}_G C$.

Now we will prove that if (i) and (ii) hold then G is 3-colorable. It suffices to prove that if G satisfies (i) and (ii) then every finite subgraph of G is 3-colorable. Let F be a finite subgraph of G. W.l.o.g. one can assume that F is biconnected. If not, F is a subgraph of a certain biconnected graph $G(W_i)$ defined in the proof of the previous theorem. Now we can use lemma 5 to get a proper edge-side coloring of F and finish the proof using Theorem 3. □

5 Conversion algorithms for 3-coloring and edge-side coloring

We have showed in Section 3 that given a 3-coloring of a plane graph G one can construct an even triangulation H including G and use it to obtain a proper edge-side coloring of G. The number of vertices in the triangulation H is $|V(H)| = O(|V(G)|)$. We have also showed that the reverse process is possible. As the proofs were constructive, they can be easily transformed into linear time algorithms converting 3-coloring into a proper edge-side coloring and vice versa. In fact these algorithms can be simplified avoiding the middle step in which one computes an even triangulation. In this section we will provide such algorithms.

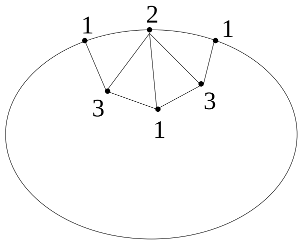
5.1 Conversion from 3-coloring to edge-side coloring

The input to the algorithm is a biconnected plane graph G (graph and its division into faces) and a proper 3-coloring $\mathcal{K}$ of G where $\mathcal{K}(v)$ is a color of vertex v. The algorithm `3COL_TO_ESC` works as follows:

ALGORITHM `3COL_TO_ESC`::
for each face f of G with the facial cycle C **do**
  Let $v_0, v_1, \ldots v_k$ be a sequence of vertices of C, in clockwise order,
  starting from an arbitrary vertex
  $m((v_0 - v_1, f)) := black$
  $i := 1$
  while $i \neq 0$ **do**
  begin
    if $\mathcal{K}(v_{i-1}) = \mathcal{K}(v_{i+1})$ **then**
      $m((v_i - v_{i+1 \pmod{k+1}}, f)) := rev(m((v_{i-1} - v_i, f)))$
    {Comment: *rev* means the opposite color}
    else
      $m((v_i - v_{i+1 \pmod{k+1}}, f)) := m((v_{i-1} - v_i, f))$
    $i := i + 1 \pmod{k+1}$
  end while

Lemma 6. *The algorithm 3COL_TO_ESC computes a proper edge-side coloring of G.*

Proof. Since G is 3-colorable then, by Theorem 2, there exists an even triangulation $H \supseteq G$ and one can extend the 3-coloring $\mathcal{K}$ of G to H. Let f be an arbitrary face of G and let C_f be its facial cycle. Observe that for arbitrary three successive vertices on C, say v_1, v_2, v_3, $dIn_H(C_f, v_2)$ is odd if and only if $\mathcal{K}(v_1) = \mathcal{K}(v_3)$:



It follows from Lemma 4 that the edge-side coloring m computed by the algorithm `3COL_TO_ESC` satisfies condition (i) of the definition of the proper edge-side coloring (because it corresponds with $C_f \cup \mathrm{Int}_H(C_f)$) and moreover m is equal to the edge-side coloring produced from H in the proof of Theorem 3. $\square$

5.2 Conversion from edge-side coloring to 3-coloring

The input to our algorithm is a biconnected plane graph G (graph and its division into faces) and a color of each side (e, f), *black* or *white*. As the result we get a function $\mathcal{K} : V(G) \longrightarrow \{1, 2, 3\}$ – a 3-coloring of G. The algorithm keeps a dynamic set FacesToColor of pairs of faces and vertices.

ALGORITHM `ESC_TO_3COL`::
for each face f of G **do**
  $VISITED(f) :=$ False
Let f be any face of G
Let v_1, v_2 be any successive vertices of G, in clockwise order
$\mathcal{K}(v_1) := 1$
$\mathcal{K}(v_2) := 2$
$VISITED(f) :=$ True
FacesToColor.Add $(f,\, v_2)$
while not FacesToColor.Empty **do**
  FacesToColor.Remove $(f,\, v_0)$
  $y := v_0$
  Let x be the vertex of f before y in clockwise order
  repeat
    Let z be the vertex of f after y in clockwise order
    if $m(y - x, f) = m(y - z, f)$ **then**
      $\mathcal{K}(z) := 6 - \mathcal{K}(y) - \mathcal{K}(x)$
    else
      $\mathcal{K}(z) := \mathcal{K}(x)$
    Let f' be the face incident to $y - z$ other than f.
    if not $VISITED(f')$ **then**
    begin
      FacesToColor.Add $(f',\, y)$
      $VISITED(f) :=$ True
    end
    $x := y$
    $y := z$
  until $y = v_0$
end while

Lemma 7. *The algorithm `ESC_TO_3COL` computes properly a vertex 3-coloring of G.*

Proof. Let $H \supseteq G$ be the even triangulation obtained from G in the proof of Theorem 3. Let $\mathcal{K}'$ be a 3-coloring of H. Observe that if $e_1 = (v - w)$ and $e_2 = (v - u)$ touch a common face f then $\mathcal{K}'(w) = \mathcal{K}'(u) \leftrightarrow m(e_1, f) \neq m(e_2, f)$. On the other hand $\mathcal{K}(w) = \mathcal{K}(u) \leftrightarrow m(e_1, f) \neq m(e_2, f)$. Therefore the process of coloring graph G in algorithm `ESC_TO_3COL` can be seen as the restriction of a greedy algorithm for coloring triangulation H. $\square$

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