Computing k-th Lyndon Word and Decoding Lexicographically Minimal de Bruijn Sequence

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CPM 2014
Moscow, June 18, 2014

Lyndon words

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Non-examples:

- 010011 ($010011 > 001101$)
- 001001 ($001001 = (001)^2$)

Enumerating Lyndon words

Notation:

Σ a fixed finite totally-ordered alphabet,

$\mathcal{L}$ all Lyndon words in Σ^+,

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$$\Sigma = \{0, 1\},$$
$$\mathcal{L}_6 = \{000001, 000011, 000101, 000111, 001011$$
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|Lynd(001110)| &= 6,
\end{aligned}$$

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Given a word w of length n the value $|Lynd(w)|$ can be computed in $\mathcal{O}(n^3)$ time.

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Corollary

The k-th lexicographically smallest Lyndon word in $\mathcal{L}_n$ can be computed in $\mathcal{O}(n^4 \log \sigma)$ time.

Proof.

Binary search over Σ^n. $\square$

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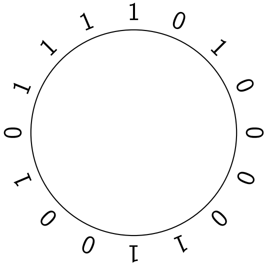
- Applications to decoding the minimal de Bruijn sequence.

De Bruijn sequences

Definition

A *de Bruijn sequence of rank n* is a cyclic sequence in which every word from Σ^n appears as a subword exactly once.



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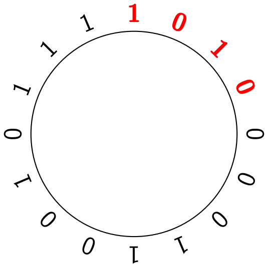

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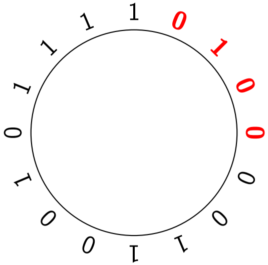

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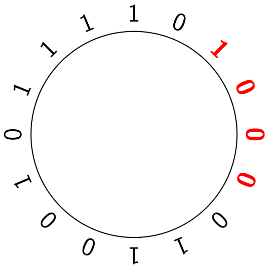

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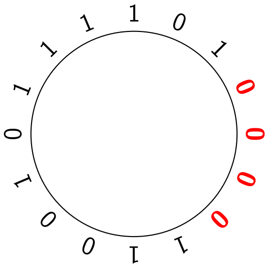

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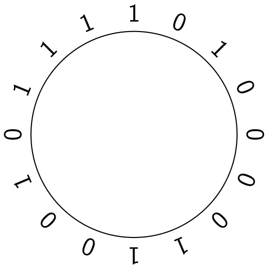
Theorem (Fredricksen, Maiorana, 1978)

The sequence dB_n is the concatenation, in lexicographic order, of all Lyndon words over Σ whose length divides n.

Decoding de Bruijn sequences

A *decoding algorithm* finds the position of an arbitrary word from Σ^n in a given de Bruijn sequence in polynomial time.



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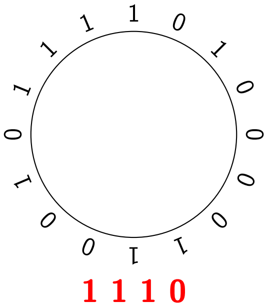

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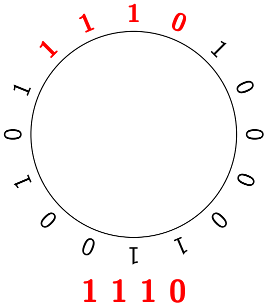

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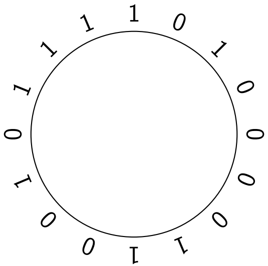

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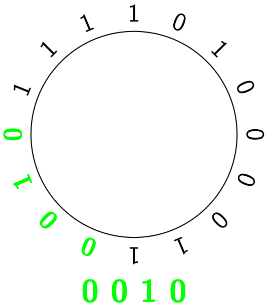

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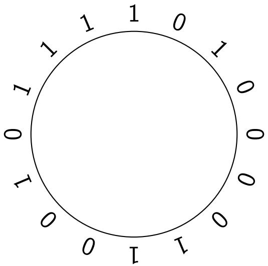

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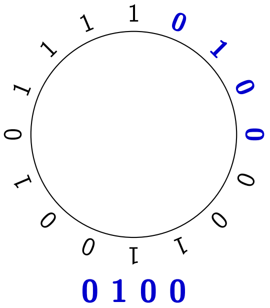

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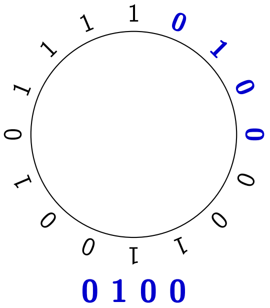

Applications:

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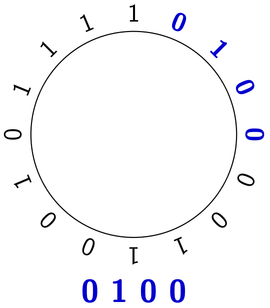

Applications:

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Previous work:

- polynomial-time decoding schemes for several types of de Bruijn sequences:
  - Paterson & Robshaw, 1995
  - Mitchell, Etzion and Paterson, 1996
  - Tuliani, 2001

Our results (minimal de Bruijn sequences)

Using the correspondence between Lyndon words and lex. min. de Bruijn sequences we show the following:

Theorem

There exists an $\mathcal{O}(n^3)$-time decoding algorithm for dB_n.

Theorem

For any n the k-th symbol of dB_n can be computed in $\mathcal{O}(n^4 \log \sigma)$ time.

Main algorithm

We sketch the proof of the following result:

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For any word w of length n, the value $|Lynd(w)|$ can be computed in $poly(n)$ time.

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For a word $w \in \Sigma^+$, denote the lexicographically minimal cyclic rotation of w by $\langle w \rangle$.

Lemma

For any word $w \in \Sigma^n$ the lexicographically maximal $w' \in \Sigma^n$ such that $\langle w' \rangle = w' \leq w$ can be found in $\mathcal{O}(n^2)$ time.

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$Lynd(w') = Lynd(w)$, so we can assume that $\langle w \rangle = w$.

Formula for $|Lynd(w)|$

Define

$$CS(w) = \{x \in \Sigma^{|w|} : \langle x \rangle \leq w\}.$$

Lemma

If $\langle w \rangle = w$ *then*

$$|Lynd(w)| = \tfrac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d})\, |CS(w[1..d])|$$

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Example. Let $w = 010111$.

$$CS(w[1..1]) = \{0\}, \quad CS(w[1..2]) = \{00, 01, 10\},$$

$$CS(w[1..3]) = \{000, 001, 010, 100\}, \quad |CS(w)| = 54$$

$$|Lynd(w)| = \tfrac{1}{6} \cdot \big(\mu(1)\,|CS(w)| + \mu(2)\,|CS(w[1..3])| + \mu(3)\,|CS(w[1..2])| + \mu(6)\,|CS(w[1..1])|\big) = \tfrac{1}{6} \cdot (54 - 4 - 3 + 1) = 8.$$

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Define a language $L(w)$ as follows:
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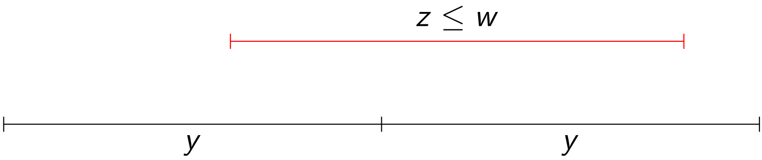

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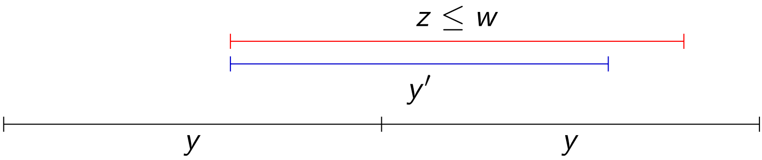
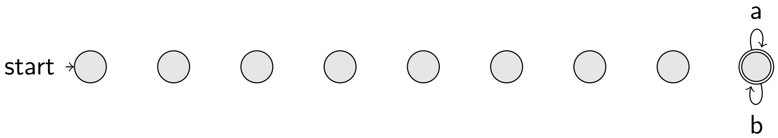


$y^2 \in L(w)$, $|y| = n$ $\qquad$ $y' \leq w$, hence $y \in CS(w)$

Deterministic automaton recognizing $L(w)$

A has $n+1$ states: one for each proper prefix of w, and an auxiliary accepting state AC. The transitions are defined as follows: $\delta(AC, c) = AC$ for any $c \in \Sigma$ and

$$\delta(w_{(i)}, c) = \left\{ \right.$$



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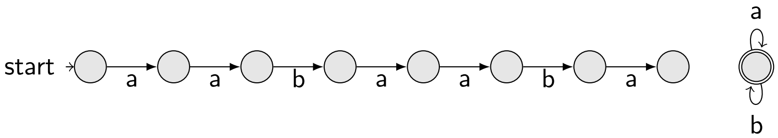

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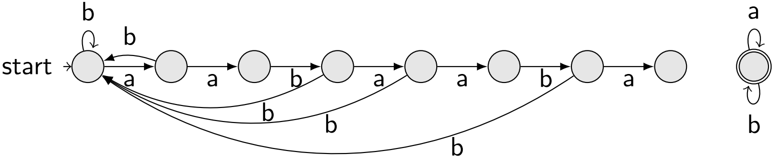

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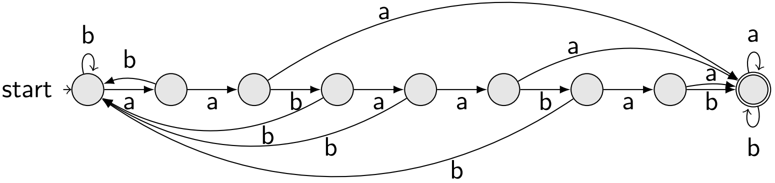

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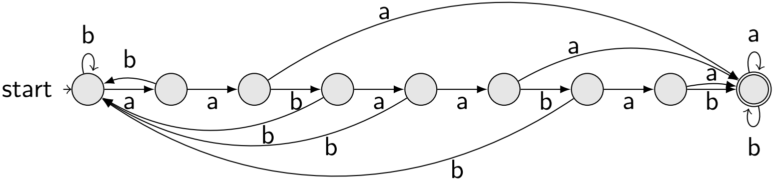

Fact

Let a word w be its own minimal rotation, i.e. $\langle w \rangle = w$. If $w[1..j]$ is a border of $w[1..i]$, then $w[j+1] \leq w[i+1]$.

Dynamic programming

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To obtain $\mathcal{O}(n^3)$ time complexity, we use an alternative method that exploits the structure of A.

Decoding de Bruijn Sequence

Let $dB_n = \lambda_1 \dots \lambda_p$, where $\lambda_i \in \mathcal{L}$ and $|\lambda_i| \mid n$.

The proof of theorem of Fredricksen and Maiorana provides, for each $w \in \Sigma^n$, λ_k such that w is a subword of $\lambda_{k-1}\lambda_k\lambda_{k+1}$.

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2. Perform pattern matching for w in $\lambda_{k-1}\lambda_k\lambda_{k+1}$.
3. The occurrence of λ_k in dB_n *ends* at position $|CS(\lambda_k^{n/|\lambda_k|})|$.

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The proof of theorem of Fredricksen and Maiorana provides, for each $w \in \Sigma^n$, λ_k such that w is a subword of $\lambda_{k-1}\lambda_k\lambda_{k+1}$.

1. Find λ_{k-1} and λ_{k+1} (using the FKM algorithm).
2. Perform pattern matching for w in $\lambda_{k-1}\lambda_k\lambda_{k+1}$.
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$\left|CS\left(\lambda_k^{n/|\lambda_k|}\right)\right|$ can be computed in $\mathcal{O}(n^3)$ time which yields the $\mathcal{O}(n^3)$-time decoding algorithm.

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- Improve the running time of the algorithms.
    - replace dynamic programming over A
      by Fast Fourier Transform in the computation of $CS(w)$.

Thank you for your attention!