1 Lower Bound

We show a family of binary words which yields a lower bound of $\frac{n}{k-1} - 3\sqrt{n}$ for the number of different factors which are k-th powers ($k \geq 2$ integer).

For $i \geq 1$ denote

$$q_i^{(k)} = (\mathtt{10}^i)^{k-1}.$$

Let $r_m^{(k)}$ be the concatenation

$$r_m^{(k)} = q_1^{(k)} q_2^{(k)} \dots q_m^{(k)} \mathtt{10}^m.$$

E.g., for squares we obtain the family of words:

$$\mathtt{1010},\ \mathtt{10100100},\ \mathtt{1010010001000},\ \mathtt{1010010001000010000}, \dots$$

and for cubes:

$$\mathtt{101010},\ \mathtt{1010100100100},\ \mathtt{1010100100100010001000}, \dots$$

Lemma 1. *The length of* $r_m^{(k)}$ *is* $(k-1)\left(\frac{m^2}{2} + \frac{3m}{2}\right) + m + 1$.

Proof. Clearly $q_i^{(k)}$ contains $k(i+1)$ bits, so

$$|r_m^{(k)}| \;=\; m + 1 + \sum_{i=1}^{m}(k-1)(i+1) \;=\; (k-1)\left(\frac{m^2}{2} + \frac{3m}{2}\right) + m + 1. \quad \square$$

Lemma 2. *The word* $r_m^{(k)}$ *contains at least*

$$\frac{m^2}{2} + 3\frac{m}{2} + \left\lfloor \frac{m+1}{k} \right\rfloor$$

different k-th powers.

Proof. Note that the concatenation $\mathtt{0}^{i-1} q_i^{(k)} \mathtt{10}^i$ contains i cyclic rotations of the k-th power $(\mathtt{0}^i\mathtt{1})^k$ distinct from this word. Apart from that, in $r_m^{(k)}$ there are $\left\lfloor \frac{m+1}{k} \right\rfloor$ k-th powers of the form $\mathtt{0}^k, \mathtt{0}^{2k}, \dots$ Thus we obtained

$$\sum_{i=1}^{m} i + \left\lfloor \frac{m+1}{k} \right\rfloor \;=\; \frac{m^2}{2} + \frac{m}{2} + \left\lfloor \frac{m+1}{k} \right\rfloor$$

k-th powers. $\square$

Theorem 1 (Lower Bound).
For infinitely many positive integers n there exists a word of length n for which the number of k-th powers is greater than $\frac{n}{k-1} - 3\sqrt{n}$.

Proof. Due to Lemmas 1 and 2, for any word $r_m^{(k)}$ we have:

$$\begin{aligned}\frac{|r_m^{(k)}|}{k-1} - \mathsf{powers}(r_m^{(k)}) &\leq \frac{m^2}{2} + \frac{3m}{2} + \frac{m+1}{k-1} - \frac{m^2}{2} - \frac{m}{2} - \left\lfloor \frac{m+1}{k} \right\rfloor \\ &< \frac{3}{2}(m+1). \end{aligned} \tag{1}$$

Obviously $\frac{3}{2}(m+1) < 3\sqrt{|r_m^{(k)}|}$, we conclude that:

$$\frac{|r_m^{(k)}|}{k-1} - \mathsf{powers}(r_m^{(k)}) < 3\sqrt{|r_m^{(k)}|} \quad \Rightarrow \quad \mathsf{powers}(r_m^{(k)}) > \frac{|r_m^{(k)}|}{k-1} - 3\sqrt{|r_m^{(k)}|}. \quad \square$$

Note. We obtain an example of a word containing $n - o(n)$ different squares that is simpler than the example by Fraenkel and Simpson [1]: we concatenate the words $q_i^{(k)} = \mathsf{1}\mathsf{0}^i$ whereas they concatenate the words $q'_i = \mathsf{0}^{i+1}\mathsf{1}\mathsf{0}^i\mathsf{1}\mathsf{0}^{i+1}\mathsf{1}$.

2 Upper Bound

Conjecture 1 *A word of length n contains less than $\frac{n}{k-1}$ different k-th powers.*

References

1. A. S. Fraenkel and J. Simpson. How many squares can a string contain? *J. of Combinatorial Theory Series A*, 82:112–120, 1998.