Modal Separation of Fixpoint Formulae

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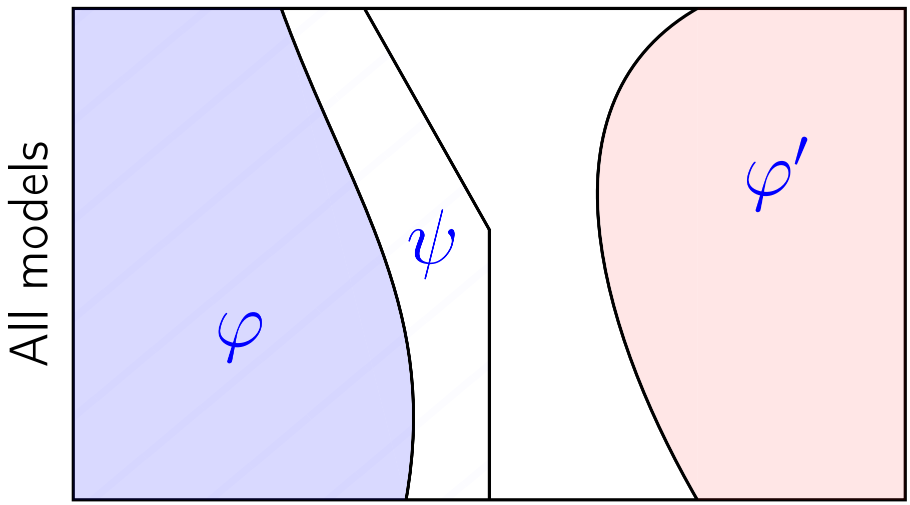
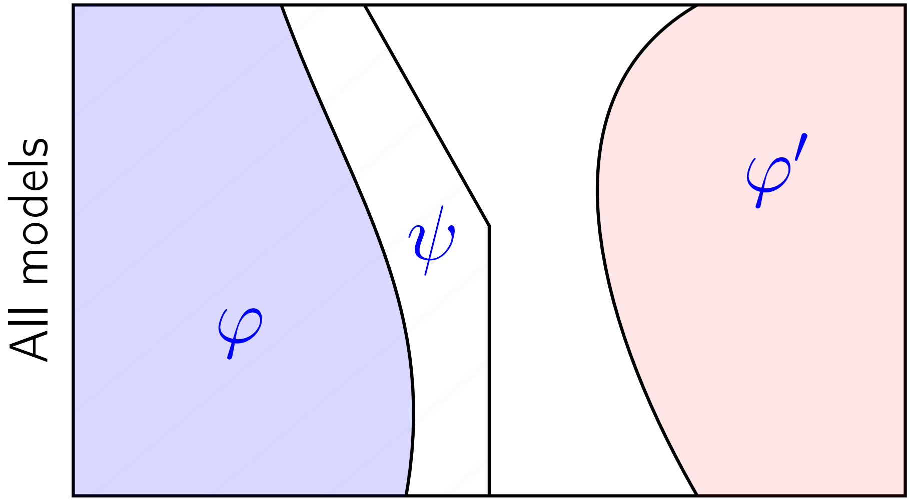
Separators



given mutually inconsistent $\varphi \models \neg\varphi'$

a **separator** is a formula ψ s.t. $\varphi \models \psi$ and $\psi \models \neg\varphi'$

Separators



in expressive logic $\mathcal{L}^+$

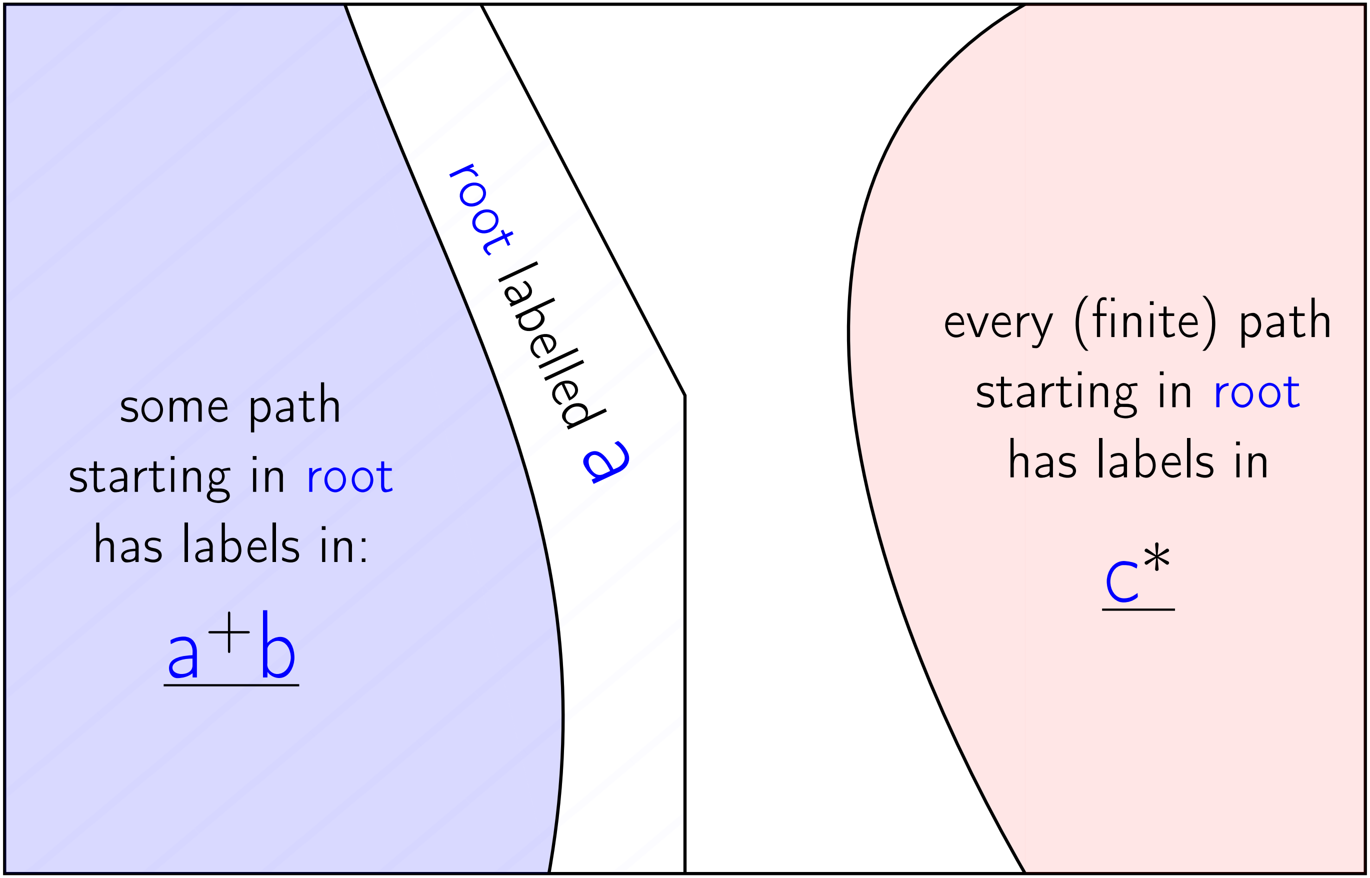
in tamed logic $\mathcal{L} \subseteq \mathcal{L}^+$

simple explanation of contradiction

complicated formulae $\varphi \models \neg\varphi'$

simple ψ s.t. $\varphi \models \psi \models \neg\varphi'$?

Example: labelled trees



Decision problem: $\mathcal{L}$-separability

given: $\varphi, \varphi' \in \mathcal{L}^+$

is there a separator $\psi \in \mathcal{L}$?

Separability generalizes definability

- For every formulae φ and ψ:

$$\psi \text{ separates } \varphi \text{ from } \neg\varphi$$

$$\Longleftrightarrow$$

$$\varphi \text{ and } \psi \text{ are equivalent.}$$

- Hence, $\mathcal{L}$-definability: "is given φ expressible in $\mathcal{L}$?"
- is a special case of $\mathcal{L}$-separability.

The logics $\mathcal{L}$ and $\mathcal{L}^+$

$\mathcal{L}$ = modal logic ML

syntax:

$$a \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid x \mid \mu x.\varphi$$

atomic propositions a $\in$ At

φ true in some child

semantics

formulae interpreted in points of labelled directed graphs

$\mathcal{L}^+ = \mu$-ML = ML + **fixpoints**

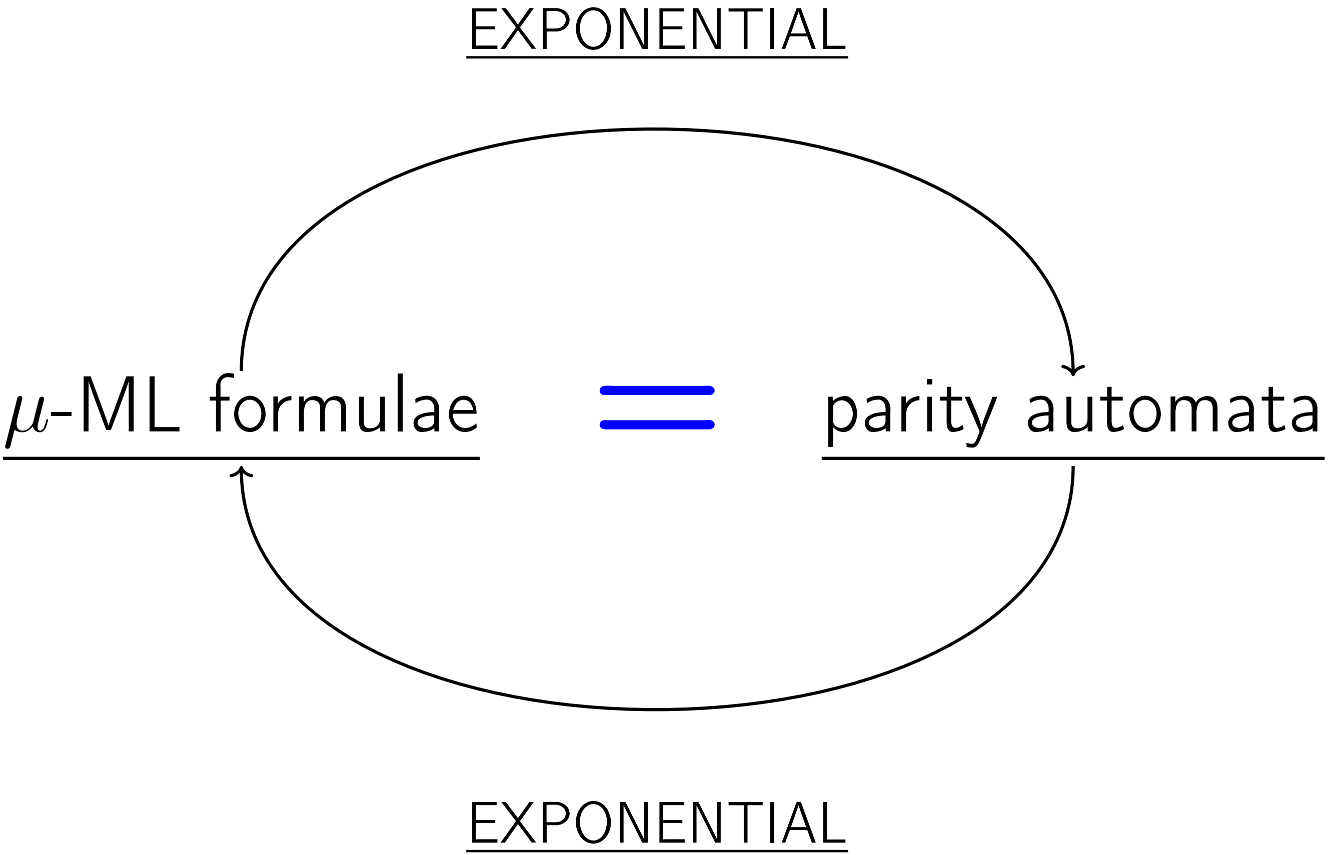
The semantics of μ-ML = ML + fixpoints

$$\mu\text{-ML}$$

$$=$$

Automata

Translations



The question: modal separability

- Given contradictory φ and φ' in μ-ML...
- ... **is there** a separator ψ in ML? Can it be **computed**?

Example

$$\varphi = \mu x.\mathsf{a} \wedge \Diamond(\mathsf{b} \vee x)$$

"some path has labels from $\mathsf{a}^+\mathsf{b}$"

$$\psi = \mathsf{a}$$

"root satisfies a"

$$\varphi' = \nu y.\mathsf{c} \wedge \Box y$$

"all (finite) paths belong to c^*"

Non-example

$$\varphi = \varphi_{\mathsf{WF}} = \mu x.\Box x$$

"no infinite paths"

φ entails no modal formulae!

$$\varphi' = \neg\varphi_{\mathsf{WF}}$$

"there is an infinite path"

The answer:

	all models	words	binary trees	d-ary trees for $d \geq 3$
ML-definability	ExpTime	PSpace	ExpTime	ExpTime
ML-separability	ExpTime	PSpace	ExpTime	2-ExpTime
separator construction	double exp.	single exp.	double exp.	triple exp.
interpolant existence for modal logic	always	always	always	coNExpTime

- all the complexity results are *completeness* results.
- *words* mean *unary trees*: words with successor relation, no order.
- in all cases trees are unordered.

What's hot:

- ML-separability is 2-ExpTime-complete over ternary trees...
- ...but only ExpTime-complete over binary trees.
- Craig interpolants (type of separators) for ML always exist over binary trees...
- ...but over ternary trees deciding its existence is coNExpTime-complete.

!!! Ternary (and higher arity) trees are harder than the binary ones! !!!

Behind separability

no modal separator for φ and φ'

$$\iff$$

for every $n \in \mathbb{N}$ there are: $\varphi \dashv \mathcal{M} \sim^{n} \mathcal{M}' \models \varphi'$

(*bisimilar* up to depth n)

- ✓ all models
- ✓ finite trees
- ✓ binary trees
- ✗ ternary trees

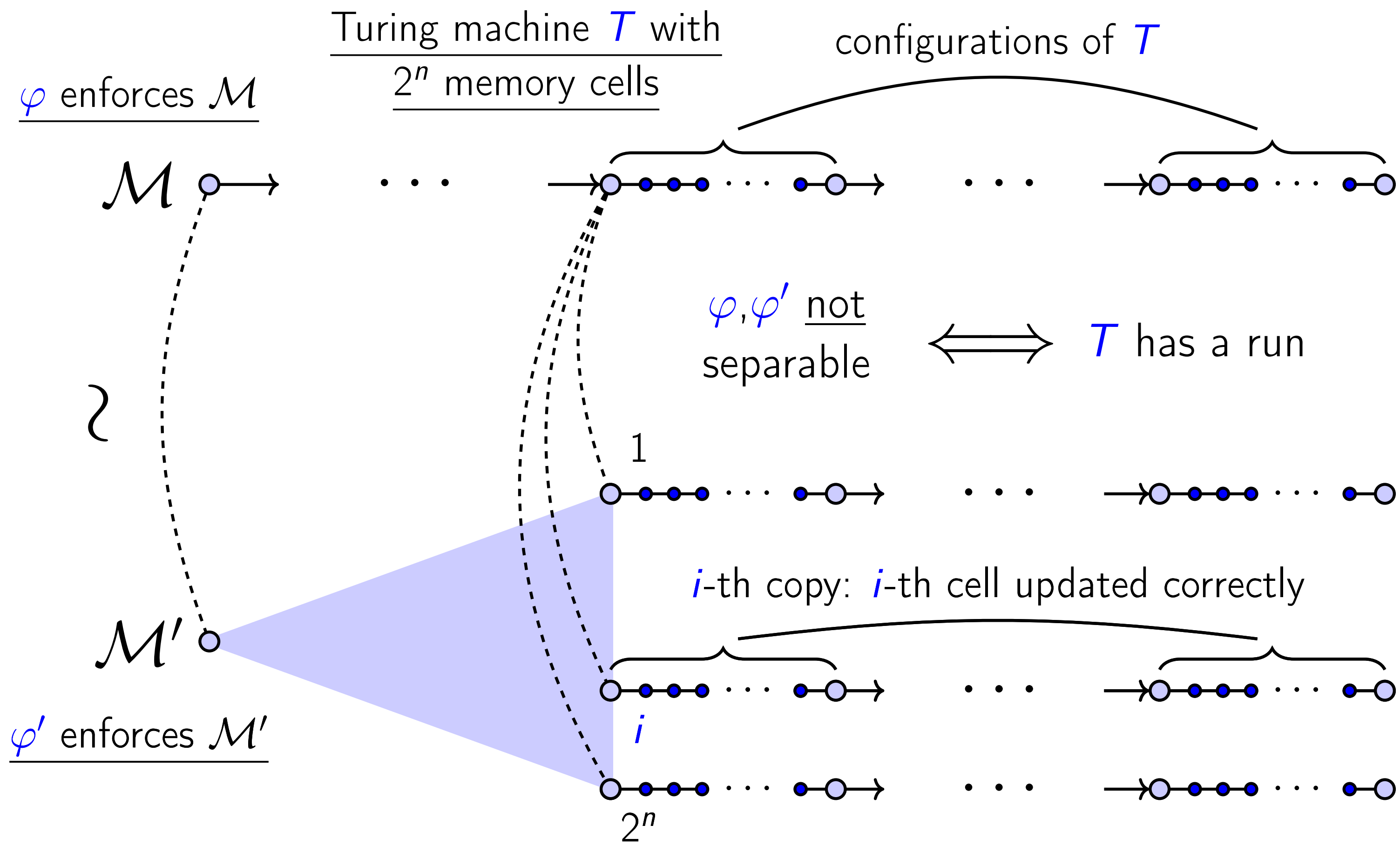
$$\overset{?}{\iff}$$

(*bisimilar* up to depth n)

for every $n \in \mathbb{N}$ there are: $\varphi \dashv \mathcal{M} \cong^{n} \mathcal{M}' \models \varphi'$

(*isomorphic* up to depth n)

Ternary case: lower bound



Ternary case: lower bound

- for a given ExpSpace Turing machine T we construct φ,φ' such that:

φ,φ' not separable $\iff$ T has a run

- idea implemented using gadgets possible over ternary, but not binary trees
- with more effort: alternating ExpSpace machines
- conclusion: modal separation is 2-ExpTime-hard over ternary trees!

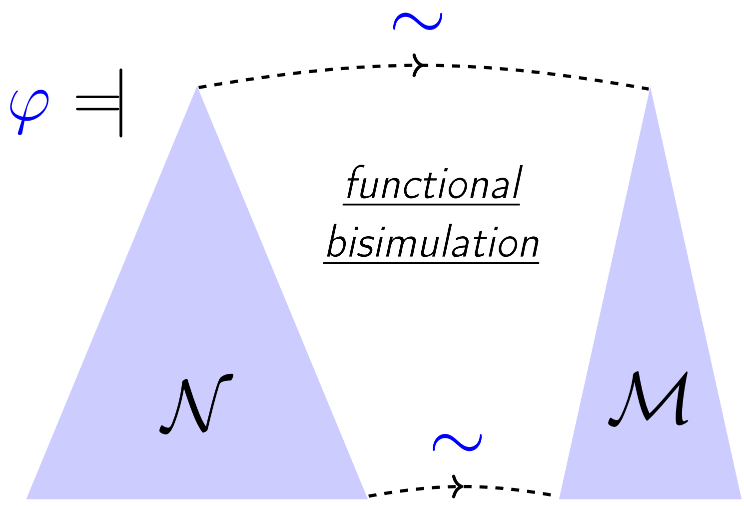
Ternary case: upper bound

- assume: tree $\mathcal{M} = (V, E)$, every node with at most ternary branching
- and a nondeterministic parity automaton $\mathcal{A} = (Q, \delta, q_I, \text{rank})$
- we define a game $\mathcal{G}(\mathcal{M}, \mathcal{A})$ played between ∃ve and ∀dam such that:

∃ve wins $\mathcal{G}(\mathcal{M}, \mathcal{A})$

$\Longleftrightarrow$

$\mathcal{M}$ is a *bisimulation quotient* of some ternary $\mathcal{N} \models \mathcal{A}$



Ternary case: upper bound

- for $\mathcal{M} = (V, E)$ and $\mathcal{A} = (Q, \delta, q_I, \text{rank})$:

$$\exists\text{ve wins } \mathcal{G}(\mathcal{M}, \mathcal{A})$$

$$\iff$$

$\mathcal{M}$ is a *bisimulation quotient*
of some ternary $\mathcal{N} \models \mathcal{A}$

- positions: $V \times Q$
- from (v,q) $\exists$ve chooses a transition $\{q_1,q_2,q_3\} = D \in \delta(q, \text{color}(v))$
- and a surjective map $h : D \to W$ where W is the set of children of v.
- $\forall$dam responds with a choice of $q_i \in D$
- the next round starts in $(h(q_i),q_i)$.
- Parity game: ranks inherited from $\mathcal{A}$.

Ternary case: upper bound

- for $\mathcal{A}$ we construct exponentially-sized $\mathcal{B}$ such that for all $\mathcal{M}$:

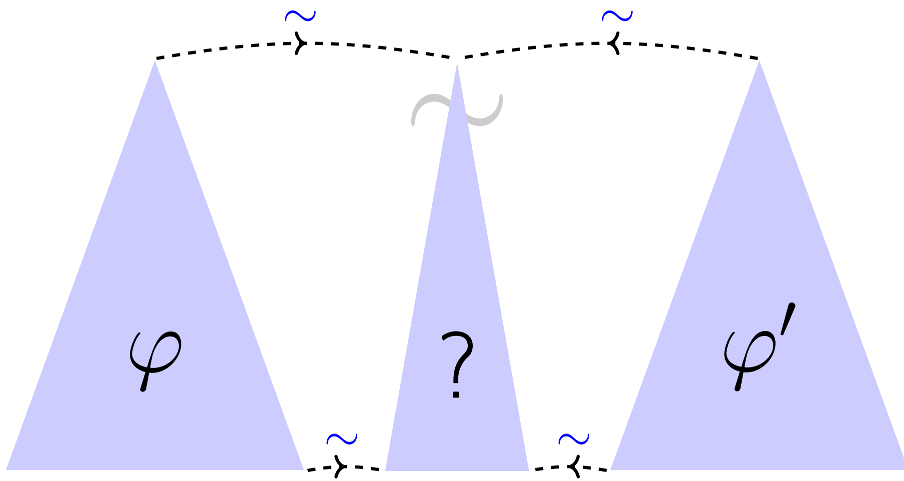
$\mathcal{B}$ accepts $\mathcal{M}$

$\iff$

$\exists$ve wins $\mathcal{G}(\mathcal{M}, \mathcal{A})$

$\iff$

$\mathcal{M}$ is a *bisimulation quotient* of some ternary $\mathcal{N} \models \mathcal{A}$

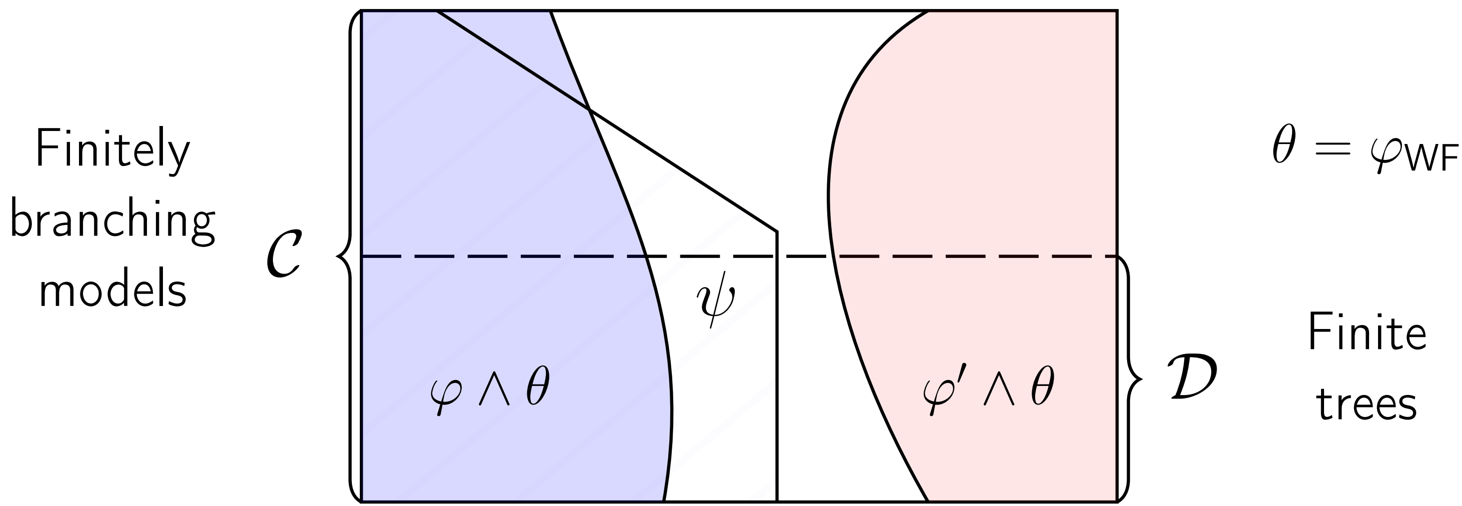


- ...and use it to decide separation.

Thank you!

Relativization

- Assume classess of models $\mathcal{C}$ and $\mathcal{D}$ and formula θ such that
- θ defines $\mathcal{D}$ in $\mathcal{C}$: $\mathcal{M} \in \mathcal{D}$ iff $\mathcal{M} \in \mathcal{C}$ and $\mathcal{M} \models \varphi$.



- Then: ψ separates φ from φ' over $\mathcal{D}$ iff it separates $\varphi \wedge \theta$ from $\varphi' \wedge \theta$ over $\mathcal{C}$.
- Example: ψ separates φ from φ' over finite words
- iff it separates $\varphi \wedge \varphi_{\mathsf{WF}}$ from $\varphi' \wedge \varphi_{\mathsf{WF}}$ over (arbitrary) words