Modal logic over the class of well-founded models is highly non-compact

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1. Introduction

Modal logic has proven useful in the fields of computer science, mathematics and philosophy. This simple yet expressive tool combines good model-theoretic and computational properties with high flexibility, allowing one to formally model wide range of diverse phenomena such as programs, proofs, graphs or even epistemic and moral belief within the same formalism ([1] is a good introduction). A natural question is which good properties of modal logic are preserved if we allow more expressive power – either by imposing non-trivial constraints (such as well-foundedness) on the models under consideration, or by enriching the language with new operators (e.g. fixed-points). We show that in both cases situation changes completely – at least when it comes to compactness.

2. Modal Logic

- Modal logic is an extension of propositional logic with an additional operator "$\Diamond$":

$$\varphi \;\mapsto\; a \mid \neg\varphi \mid \varphi \to \varphi \mid \Diamond\varphi$$

  where "a" belongs to a fixed set Σ of *atomic propositions.*
- Modal formulae are interpreted in nodes of a *model*, i.e.:
  - a directed graph (V, E) (possibly infinite) together with
  - a valuation $\sigma_v : \Sigma \to \{\mathsf{true}, \mathsf{false}\}$ for each node $v \in V$

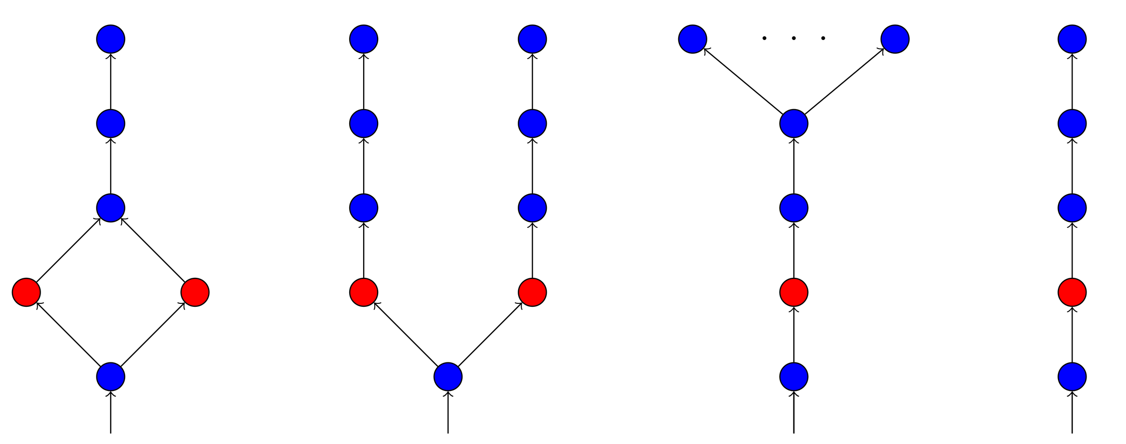
Atomic formulae are interpreted according to the valuations, and the meaning of a formula "$\Diamond\varphi$" is: "The current node has a child satisfying φ". A *rooted* model is a model with a chosen root $r \in V$.

3. Bisimulation

Bisimulation is a natural equivalence relation capturing behavioural equivalence. Given two models, *bisimilarity* is the greatest relation Z between their underlying sets of vertices s.t. whenever pZq,

- p and q satisfy the same atomic propositions,
- for every child p' of p there exists a child q' of q s.t. $p'Zq'$ – and vice versa.

Rooted models models are bisimilar iff their roots are.



4. The result

We show that modal logic interpreted over the class of all well-founded models is not κ-compact for any reasonably small cardinal κ. That is, for each such κ there exists a set T of modal formulae (over κ-many atomic propositions) s.t.:

- each subset of T of size strictly smaller than κ is satisfiable in a well-founded model, but
- the entire T is not.

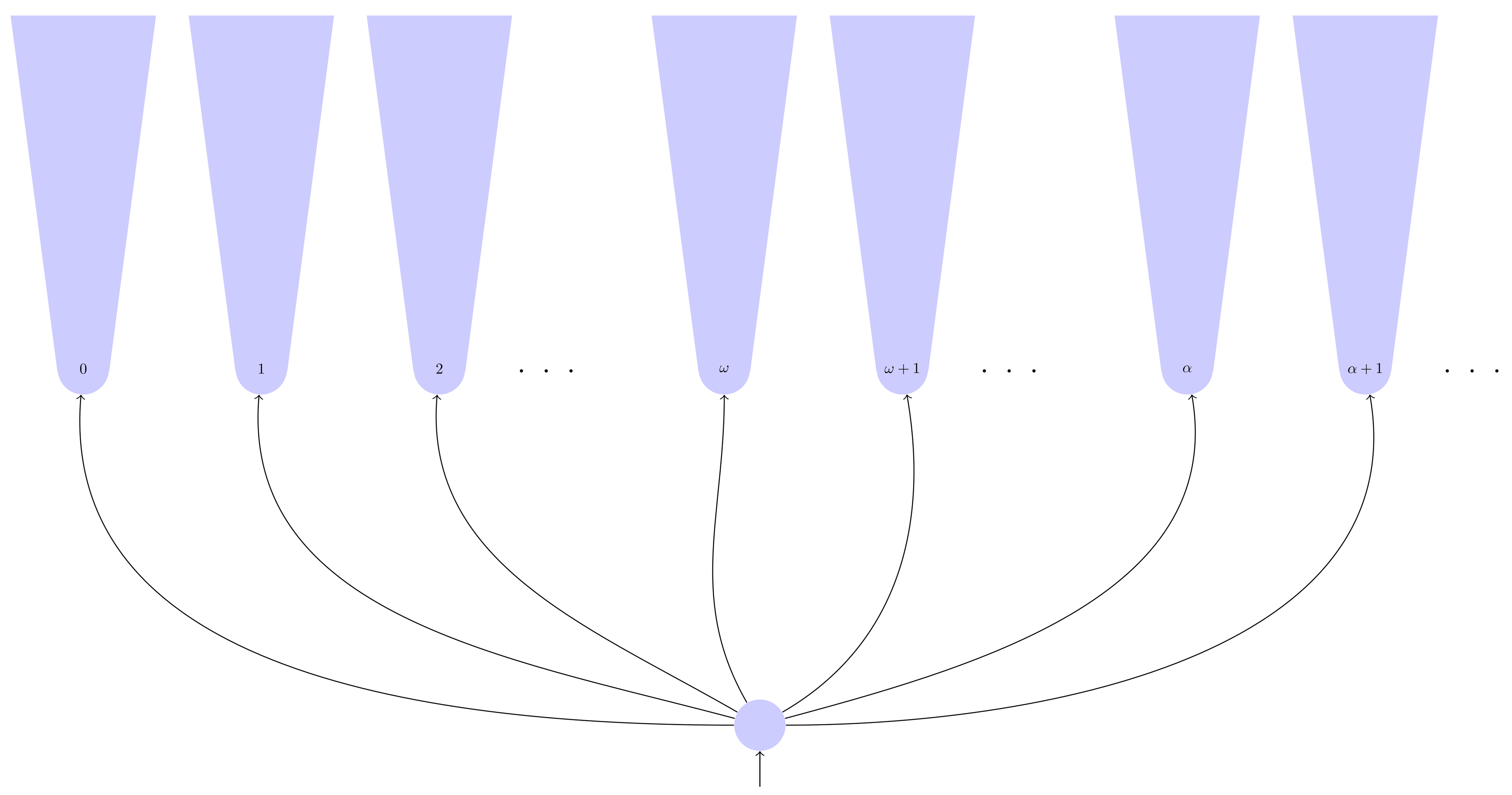
Here by "reasonably small" we mean that κ is smaller than the first cardinal equinumerous with the set of all smaller cardinals (i.e. $\kappa < \lambda$ for λ being the least solution to $\lambda = \omega_\lambda$)

5. The idea of the proof

In the proof we construct a well-founded rooted model $\mathcal{M}$ where:

- the root has κ-many children,
- each of them is definable (i.e. it is the only child satisfying certain modal formula),
- any other well-founded $\mathcal{N}$ satisfying the same modal formulae as $\mathcal{M}$ is bisimilar to $\mathcal{M}$.

We proceed by transfinite induction on the cardinal κ. The details are tedious but not hard, and the key idea is the following observation. Assume we have a model $\mathcal{M}$ as described above:



Let us enumerate the children of the root with ordinal numbers and for the α-th child pick a formula φ_α that defines it. We extend Σ with fresh atomic propositions $\{c_\delta \mid \delta < \kappa^+\}$ and require that:

- $\{\Diamond c_\delta \mid \delta < \kappa^+\}$ – each c_γ is satisfied in some child,
- $\{\neg\Diamond(c_\delta \wedge c_\gamma) \mid \delta \neq \gamma \text{ and } \delta, \gamma < \kappa^+\}$ – no child satisfies two different $c_\delta \neq c_\gamma$
- $\{\neg(\Diamond(c_\delta \wedge \varphi_\alpha) \wedge \Diamond(c_\gamma \wedge \varphi_\alpha)) \mid \delta \neq \gamma \text{ and } \alpha, \delta, \gamma < \kappa^+\}$ – bisimilar copies of α-th child cannot satisfy different $c_\delta \neq c_\gamma$.

On the one hand, any subset $\Gamma_0 \subseteq \Gamma$ of the above requirements s.t. $|\Gamma_0| < \kappa^+$ can be satisfied in a model whose reduct to Σ is bisimilar to $\mathcal{M}$ – as we need to distribute at most $|\Gamma_0|$-many new labels among κ-many nodes. However, by the pigeonhole principle this is not possible for the entire Γ – and so the modal theory of $\mathcal{M}$ together with Γ is κ-satisfiable, but not satisfiable.

6. Corollaries

- The modal fixed-point calculus μ-ML is also highly non-compact, as it can express well-foundedness.
- This also implies that no proof system with infinitary rules of small arity can be strongly complete for neither modal logic over well-founded models nor μ-ML.
- Unlike with usual modal logic (see [2]), it may be hard to characterise when a given set of μ-ML formulae has a unique model *up to bisimulation.*

7. References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic.* Cambridge University Press, 2002.

[2] Jędrzej Kołodziejski. *Bisimulational Categoricity.* to appear in Volume 13 of Advances in Modal Logic, 2020.