Countdown logic, games and automata

bisimulation-invariant approach to (un)boundedness

Jędrzej Kołodziejski
(& Bartek Klin)

4 VII 2023
eTokio

μ-calculus = modal logic + fixpoints

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plus val : Var $\to \mathcal{P}(M)$

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(note: $F_\mu^0 = \bigcup \emptyset = \emptyset$ and $F_\nu^0 = \bigcap \emptyset = M$)

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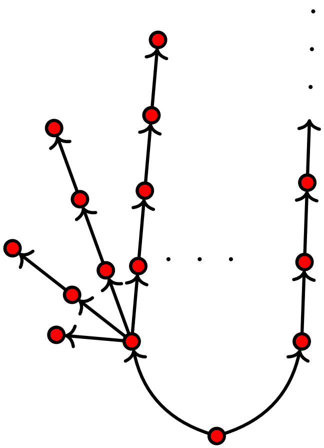

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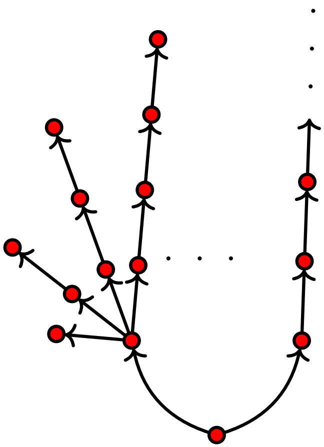

$F^0_\nu = M$ $\qquad$ $F^1_\nu = F(F^0_\nu)$

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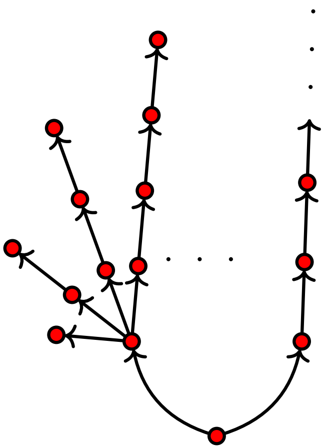

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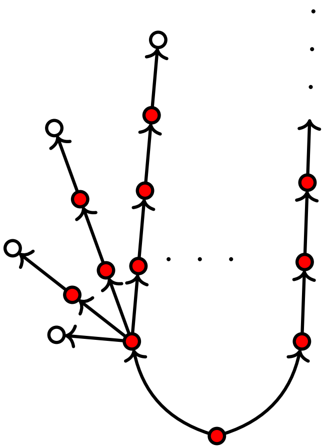

$F_\nu^1 = F(F_\nu^0) =$ "$\exists$ a child"

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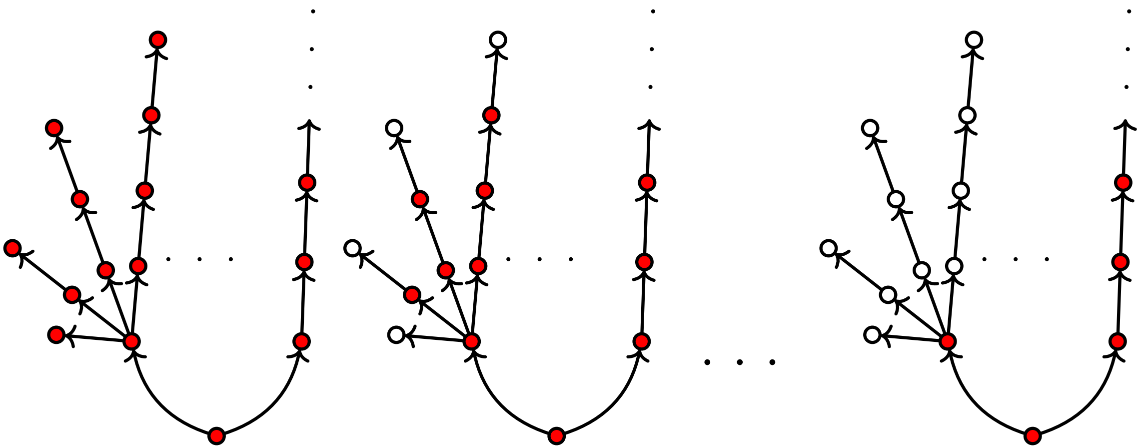

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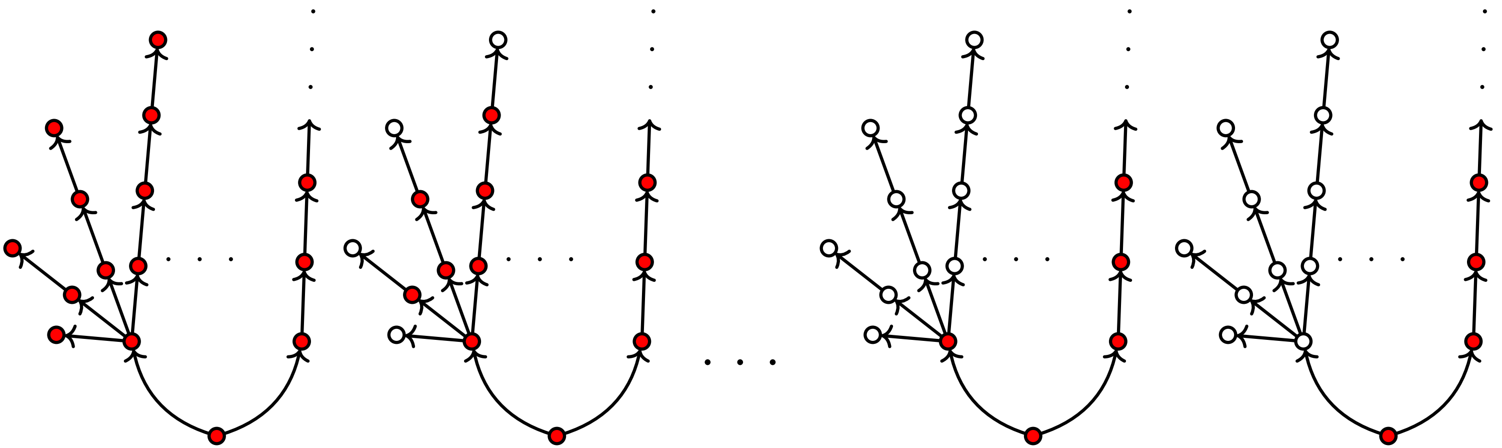

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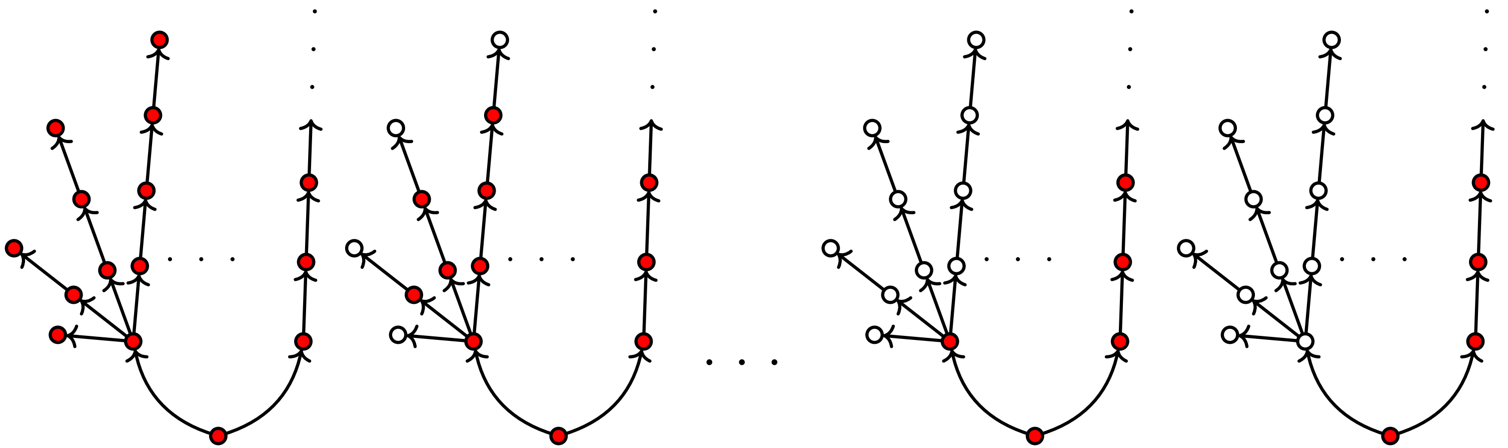

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regular languages

(of finite/infinite words, trees... or up to bisimulation)

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$V_\exists \sqcup V_\forall$

$\parallel$

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(strategies, winning strategies, etc. defined as usual)

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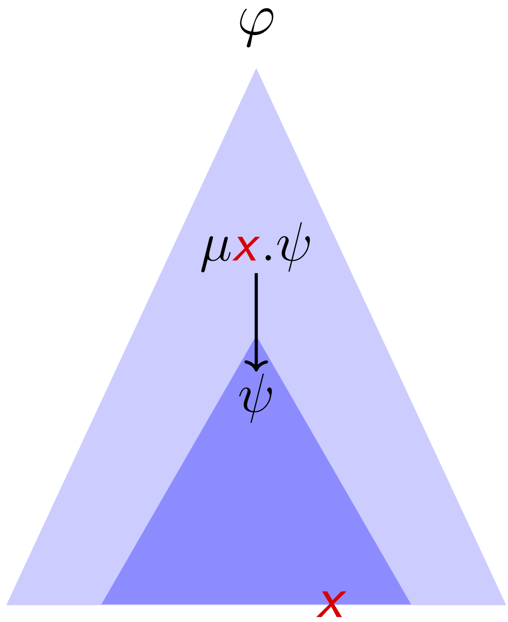

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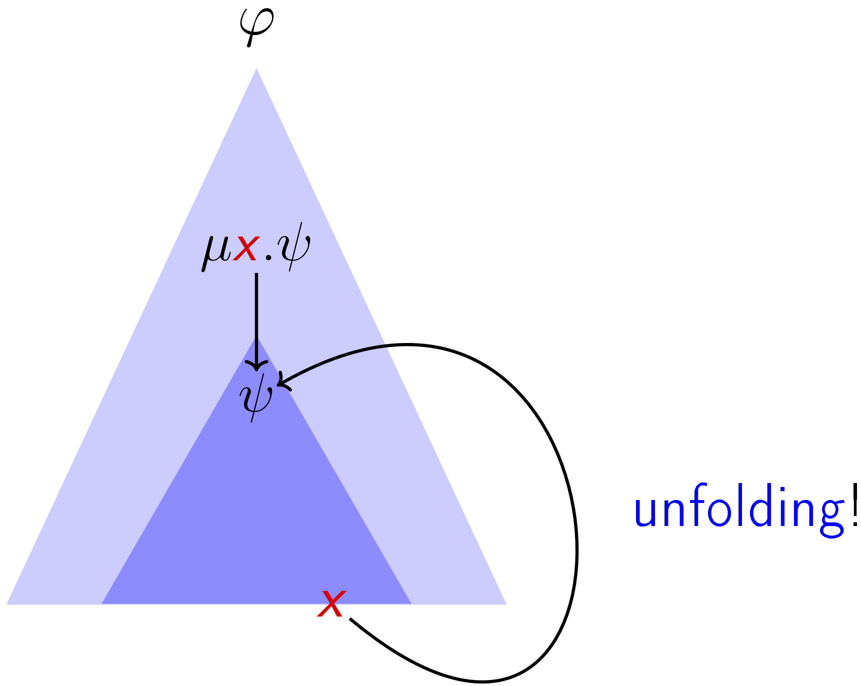

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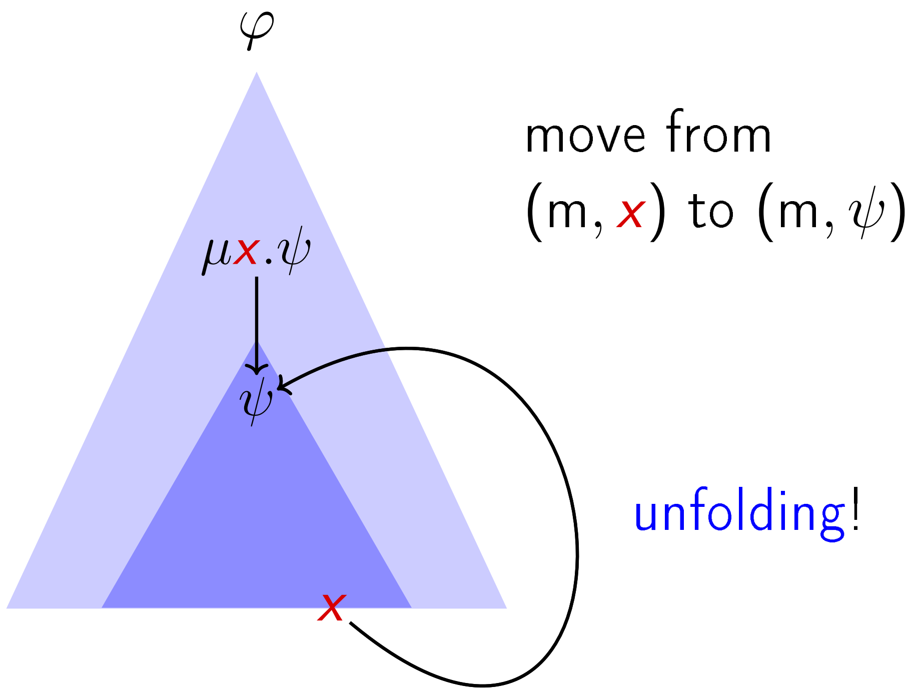

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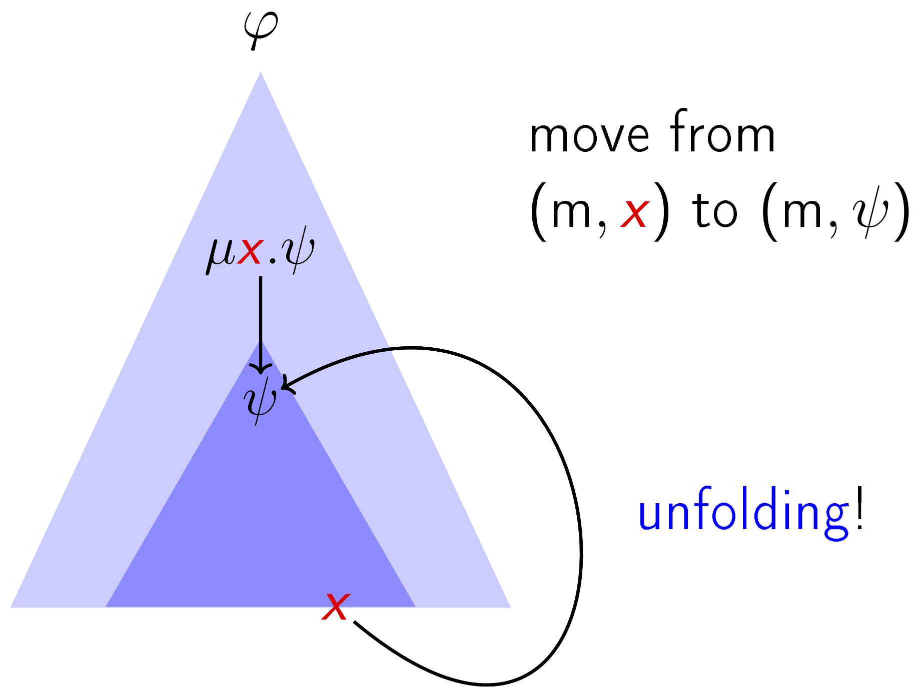

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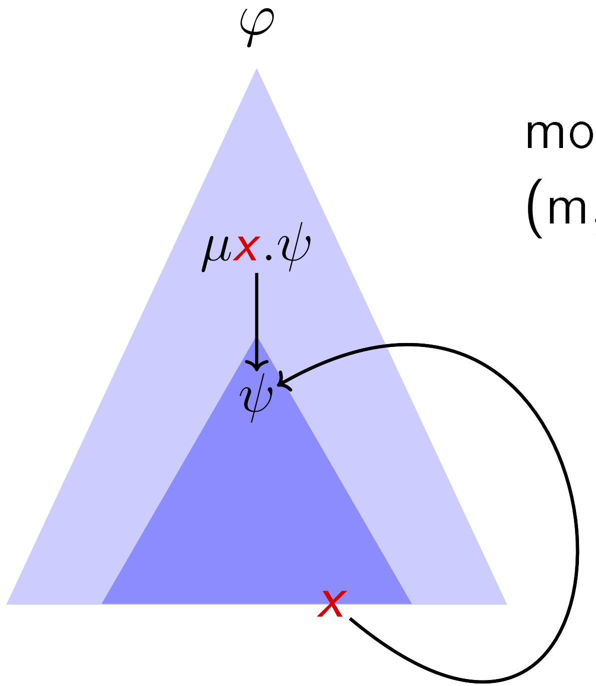

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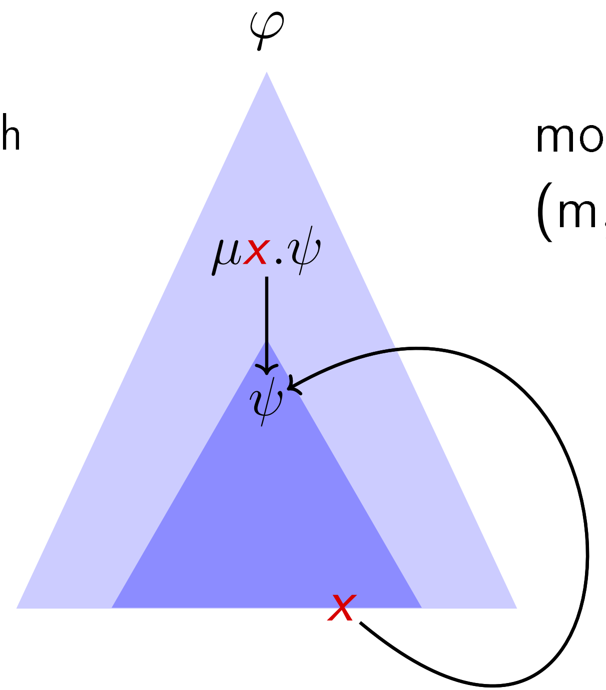

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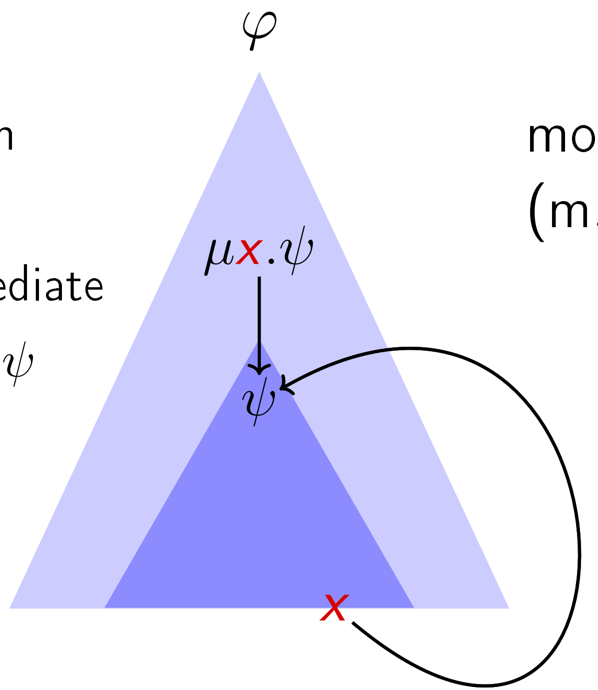

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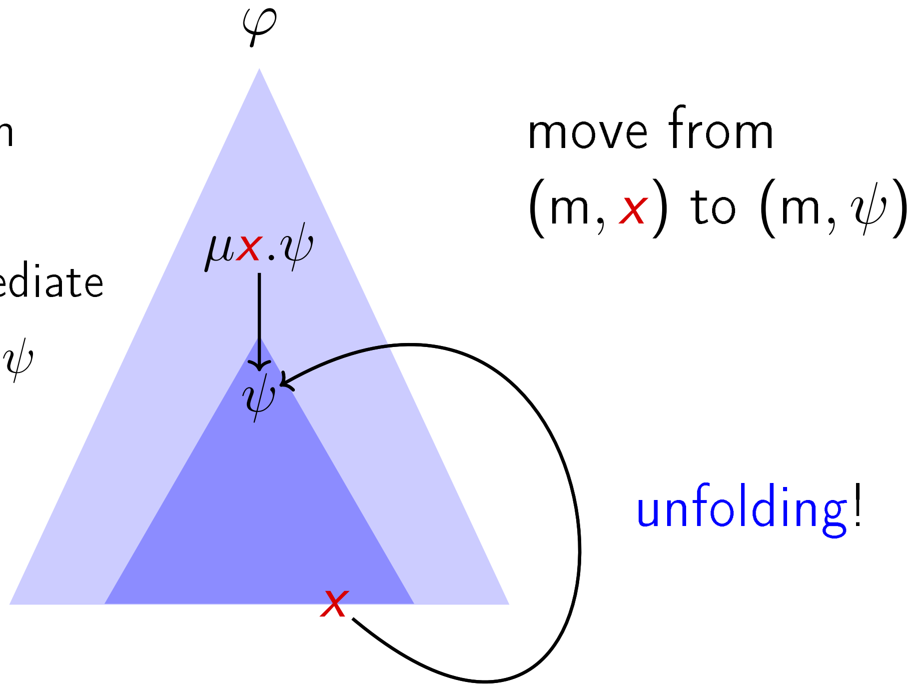

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x \arrow[u, bend right=90]
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∃ve chooses successor
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deterministic & point not changed

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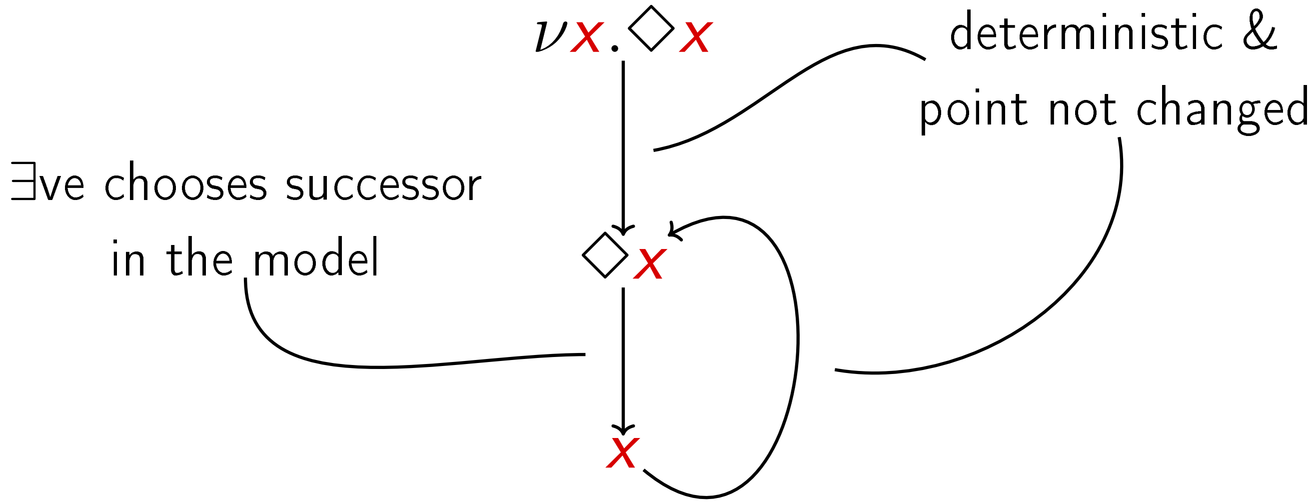

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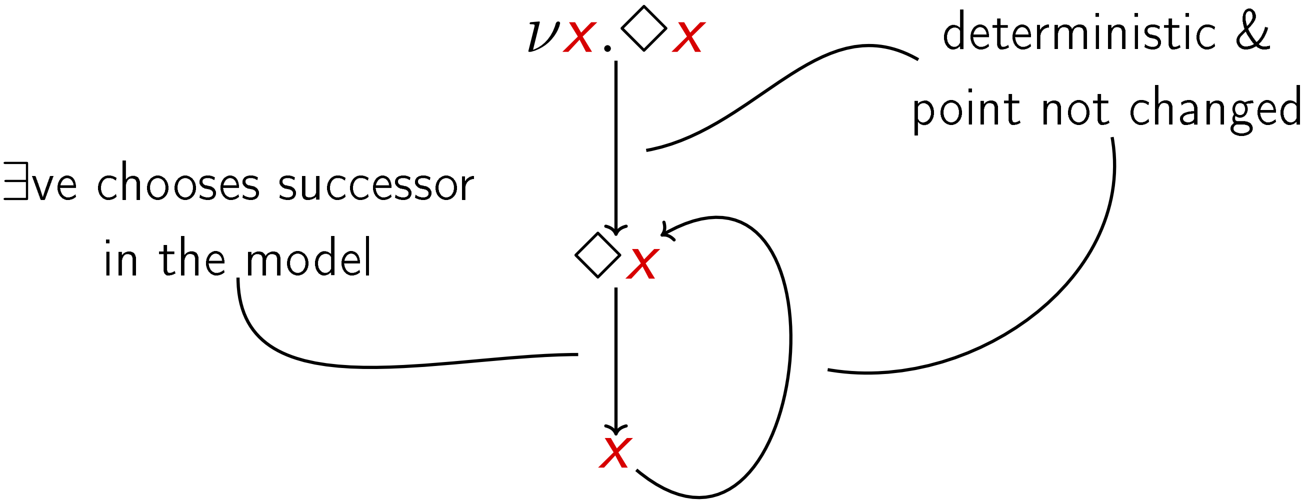

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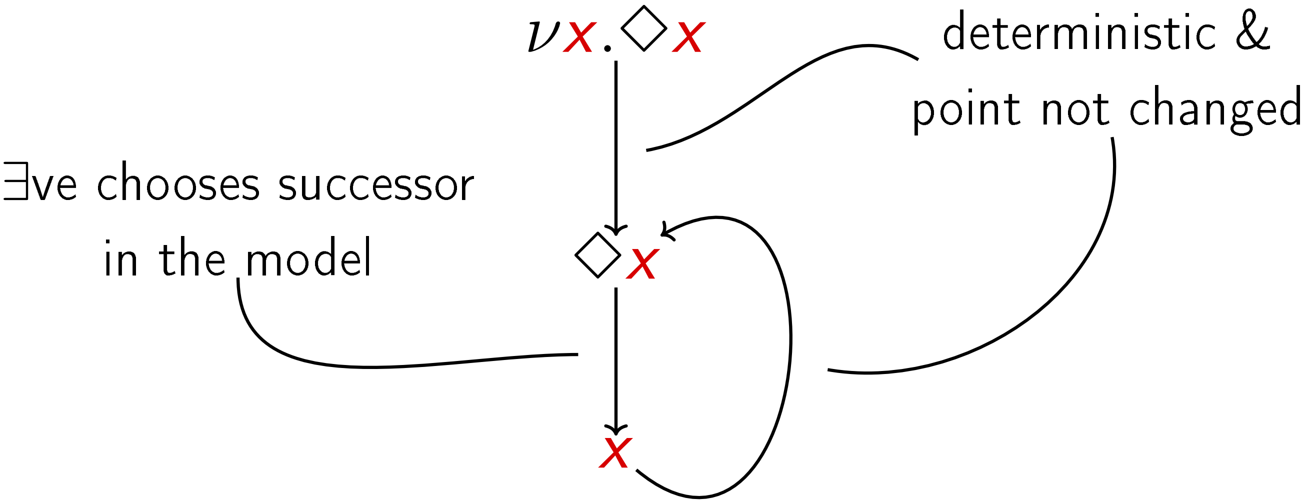

- $\exists$ve wins all infinie plays
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works for every $\mathcal{M}$ and φ!

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$\mathbin{/\!/}$

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language of $\mathcal{A}$:

$\mathcal{A}$ accepts $\mathrm{m} \in M$ $=$ $\exists$ve wins the game from (m, q_I)

$\underline{\mu\text{-ML formulae}}$ $\underline{\text{parity automata}}$

$$
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\underline{\mu\text{-ML formulae}} \arrow[rr, bend left, "\underline{\text{game semantics}}"] & & \underline{\text{parity automata}}
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game semantics

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μ-ML formulae

parity automata

μ-ML describes arbitrary automata

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$$\begin{tikzcd}[column sep=huge]
\mu\text{-ML formulae} \arrow[rr, bend left=60, "\text{game semantics}"] & = & \text{parity automata} \arrow[ll, bend left=60, "\mu\text{-ML describes arbitrary automata}"]
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we want to extend μ-ML!!!

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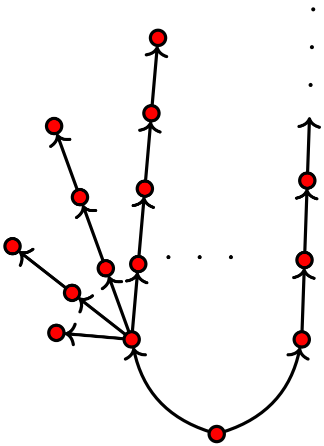

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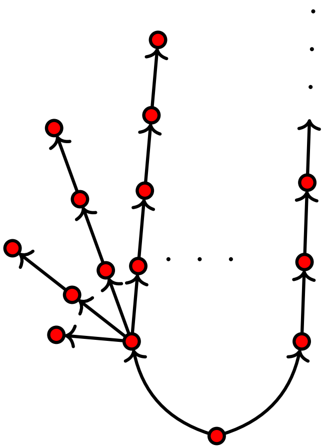

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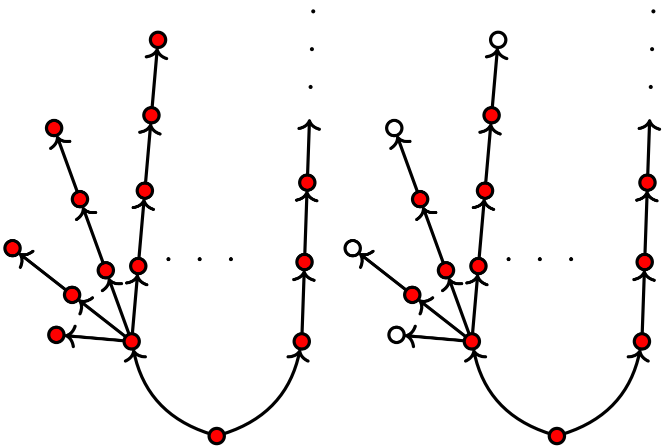

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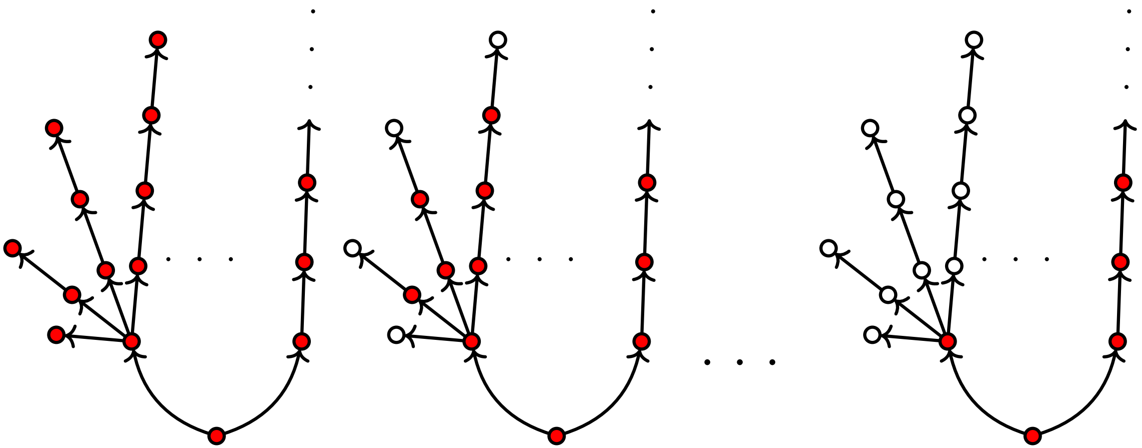

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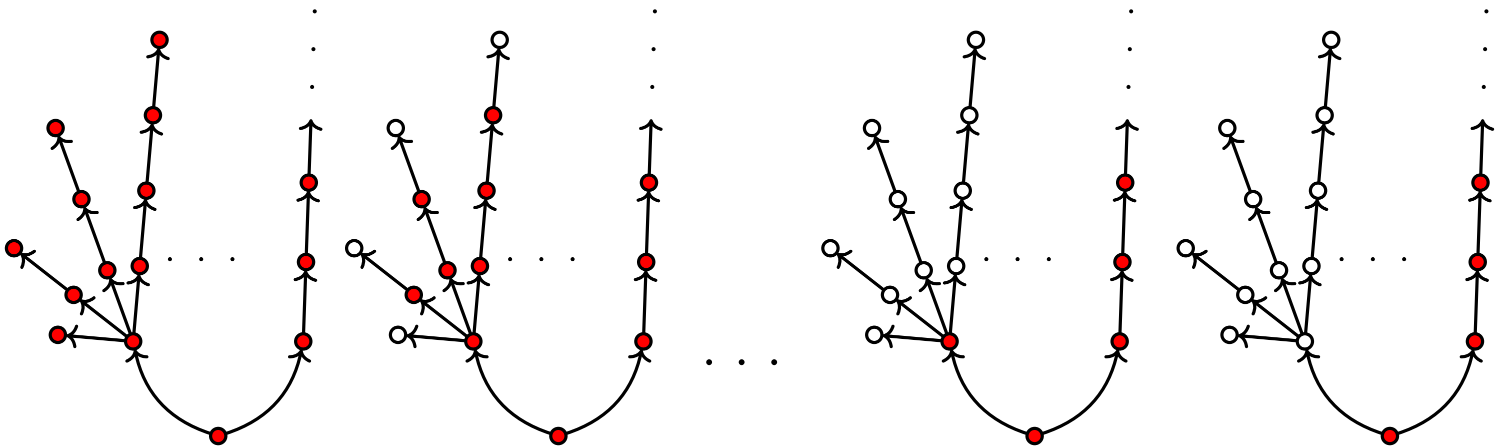

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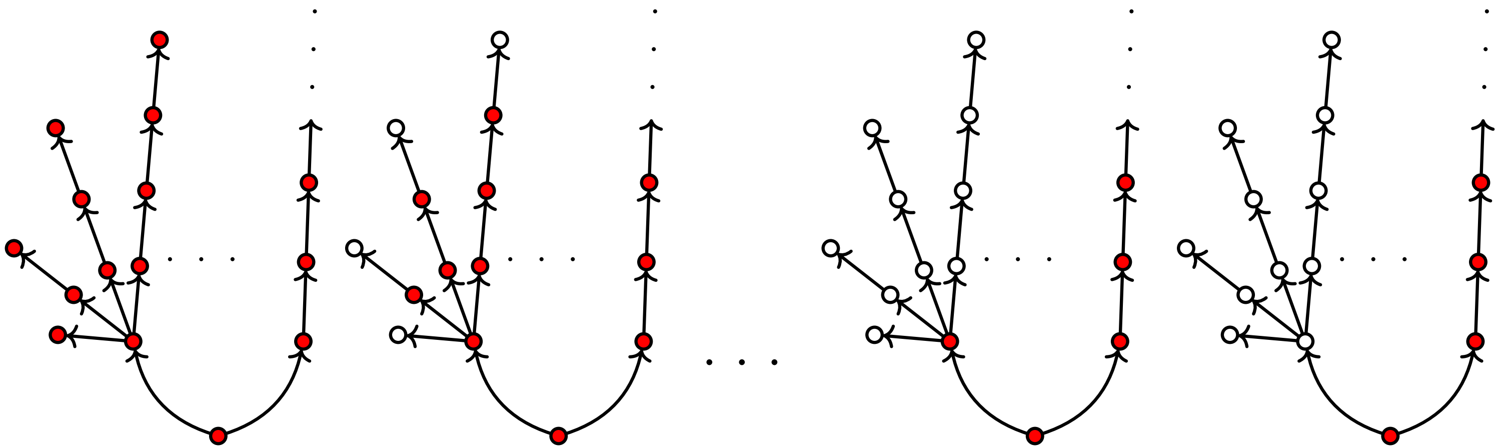

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$$\| $$

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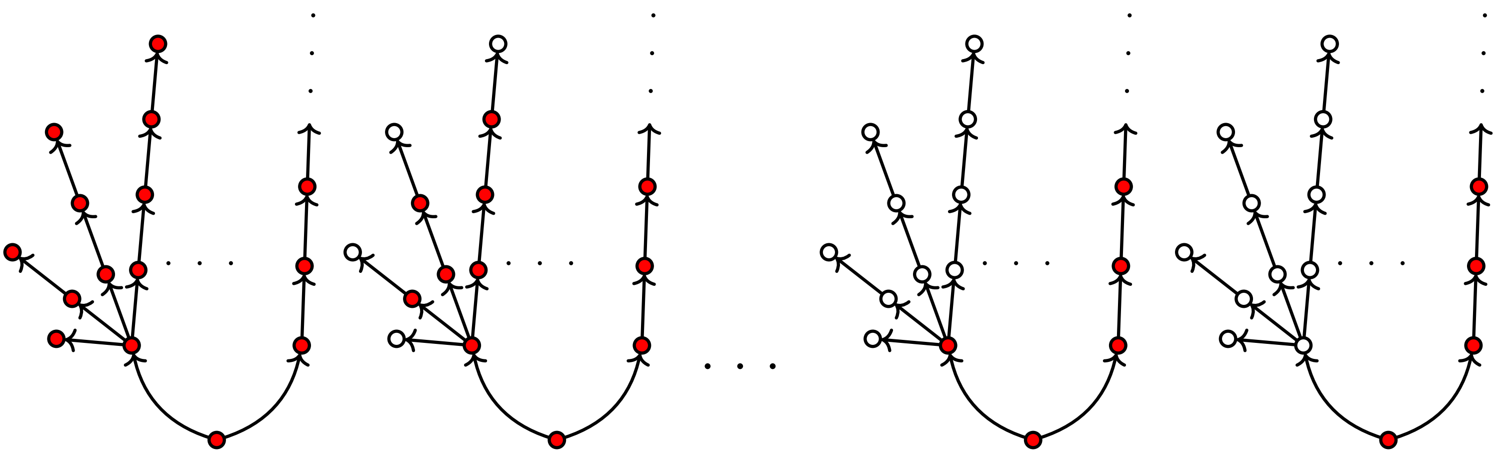

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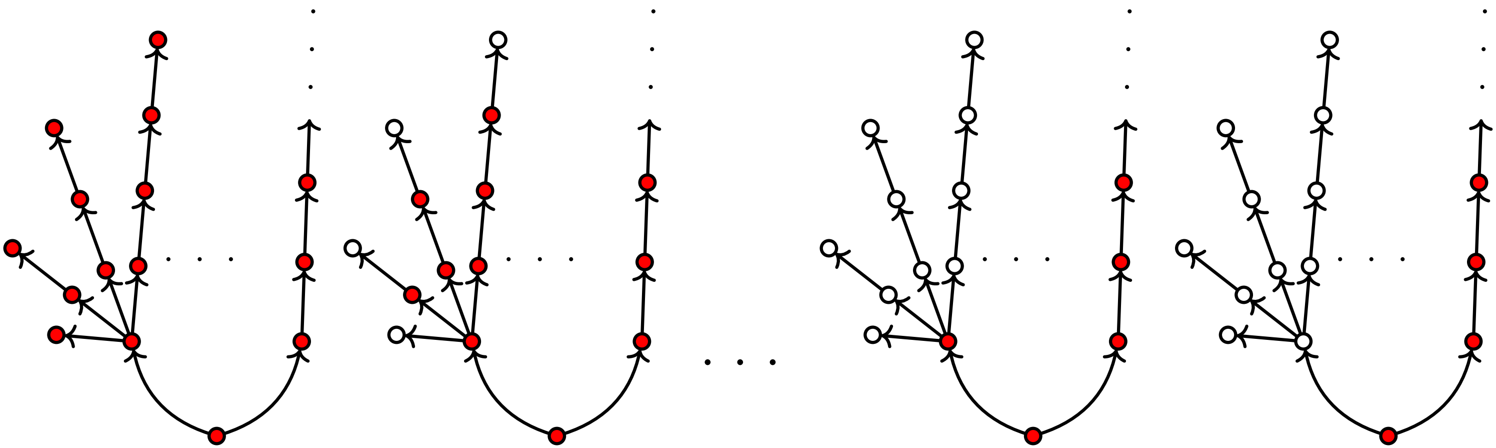

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μ-calculus + countdown operators μ^ω, ν^ω

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$$\underline{\text{countdown } \mu\text{-calculus}}$$

Syntax:

extended with μ^ω and ν^ω

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=

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$$[\![\mu^\omega x.\varphi]\!]^{\mathsf{val}} = F_\mu^\omega \quad \text{and} \quad [\![\nu^\omega x.\varphi]\!]^{\mathsf{val}} = F_\nu^\omega$$

Semantics:

μ-calculus $\sim$ parity games

equivalent to alternating parity automata

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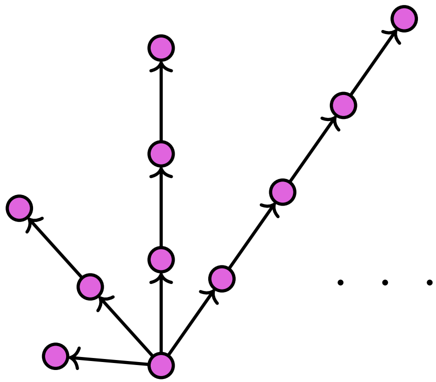

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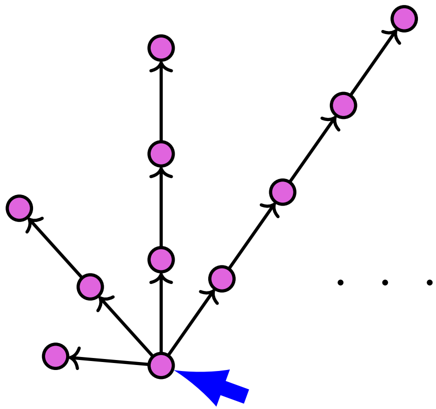

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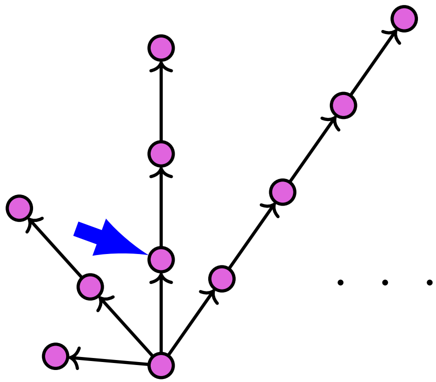

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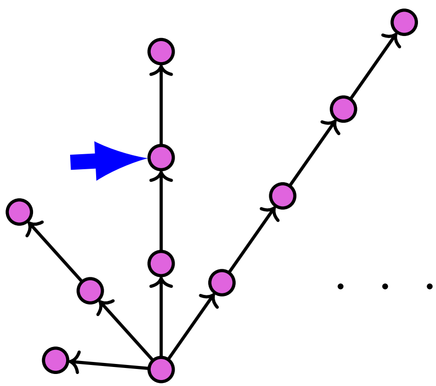

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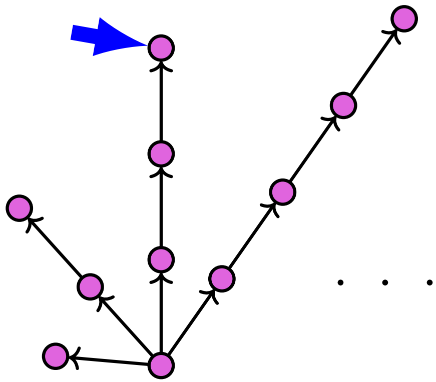

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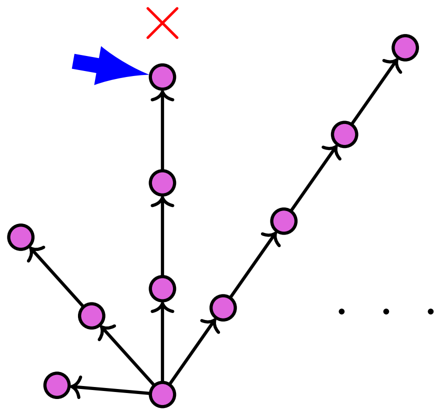

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$\Longleftrightarrow$

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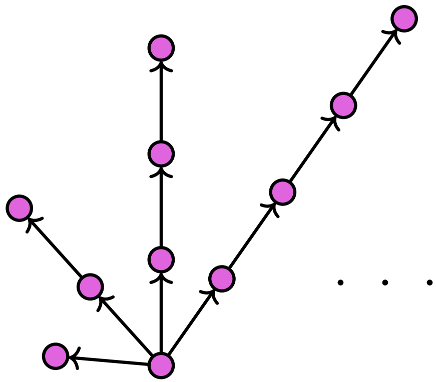

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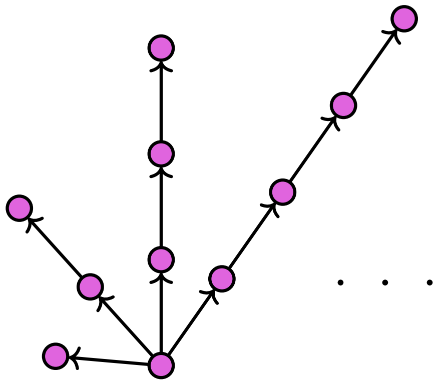

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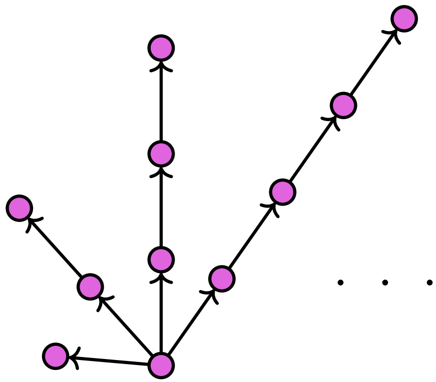

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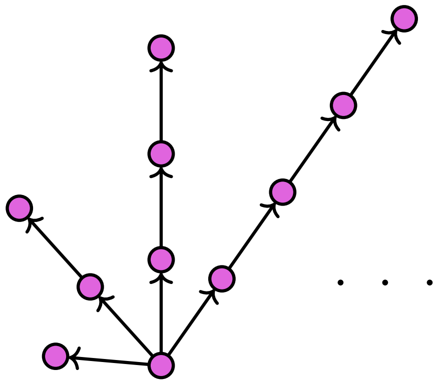

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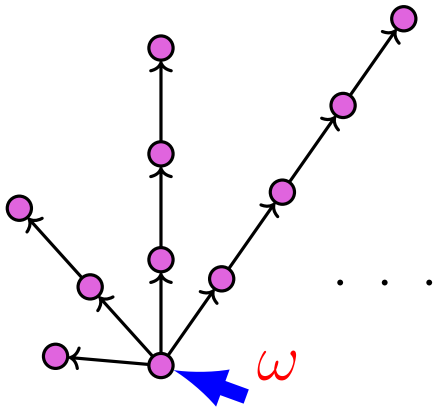

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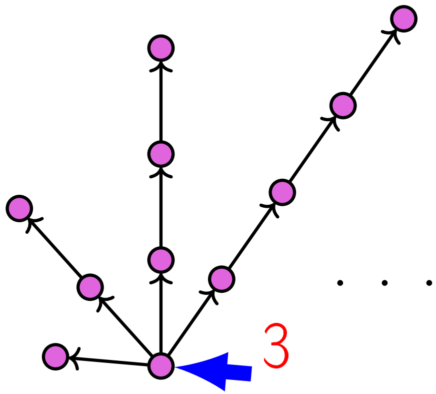

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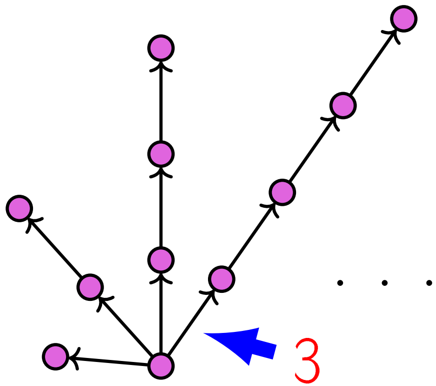

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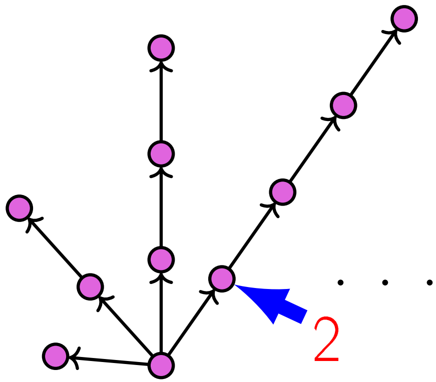

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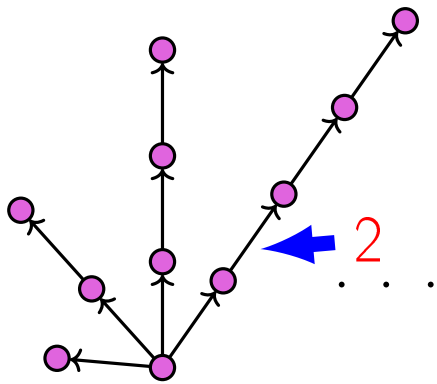

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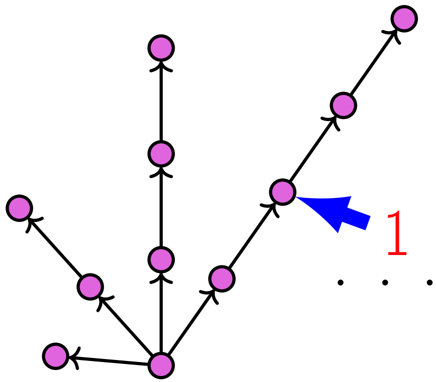

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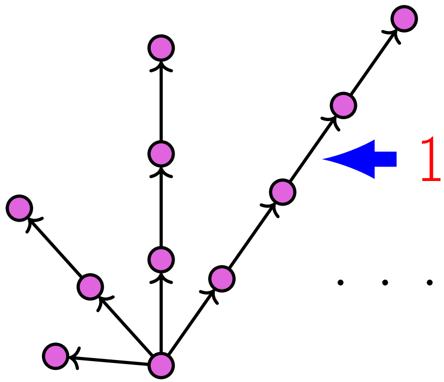

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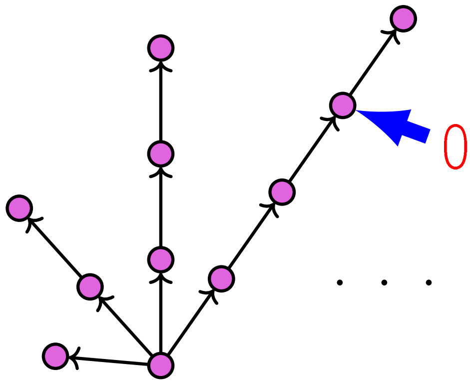

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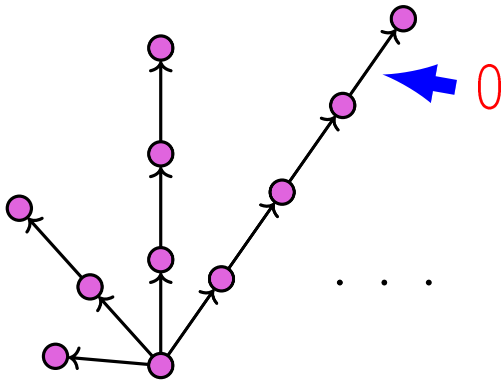

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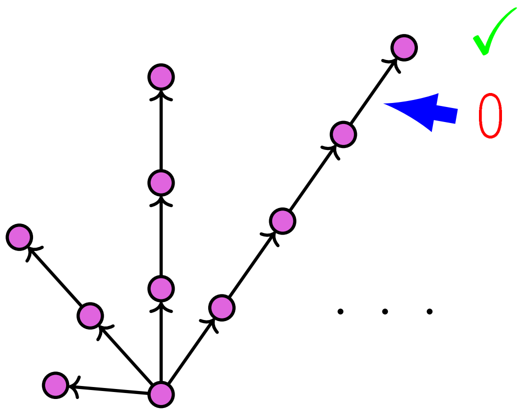

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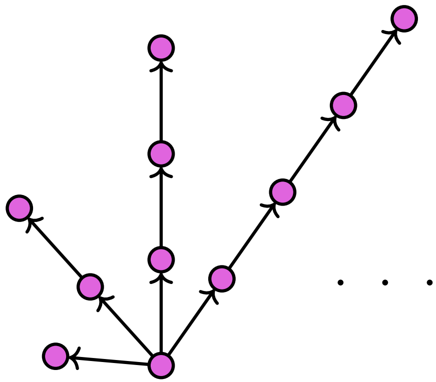

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$\Longleftrightarrow$

$\exists$ arbitrarily long paths

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- then, owner of v chooses vEw
- and the game moves to $(w, \overline{\mathsf{C}'_r})$.

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works for every $\mathcal{M}$ and φ!

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- semantic via a countdown game: the parity game for $(Q, \delta, q_I, \mathrm{rank})$ but the ranks $\mathcal{D} \subseteq \mathcal{R}$ are now nonstandard!

countdown μ-ML

countdown automata

countdown game semantics

countdown μ-ML

countdown automata

countdown game semantics

$$\begin{tikzcd} \text{countdown } \mu\text{-ML} \arrow[r, bend left] & \text{countdown automata} \arrow[l, bend left] \end{tikzcd}$$

countdown μ-ML describes arbitrary automata

countdown game semantics

countdown μ-ML = countdown automata

countdown μ-ML describes arbitrary automata

equivalent to alternating ~~parity~~ automata
countdown!

countdown μ-calculus $\sim$ ~~parity~~ games
countdown!

wining regions definable in logic

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COMPLICATIONS!!!

wining regions definable in logic

equivalent to alternating ~~parity~~ automata

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no simple nondeterministic model!!!

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but not for μ^{ω}-ML!!!

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  - Büchi countdown automata: only two ranks $r^\exists < r^\forall$, over infinite words

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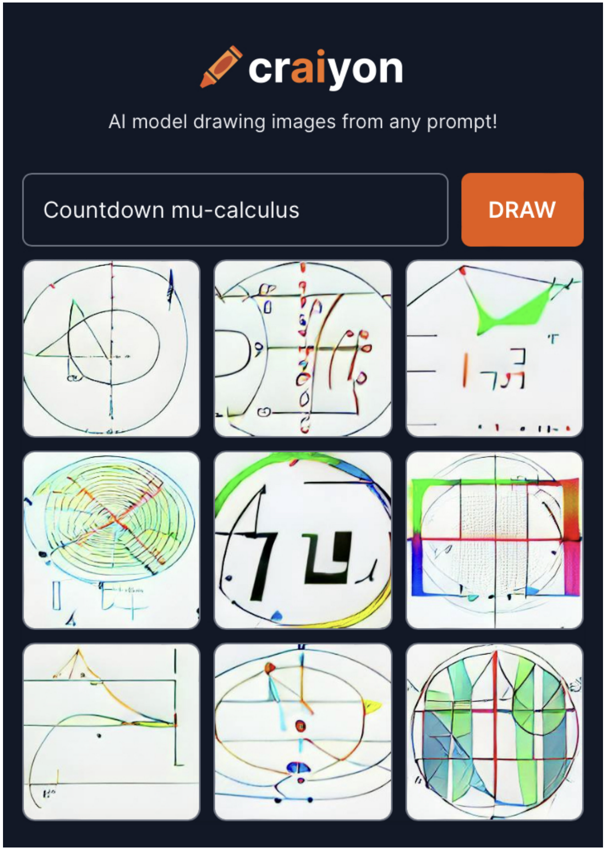
("$\Diamond$" means "supremum over children"; $\widetilde{f}(t) = \frac{1}{2}t + \frac{1}{2}$ is dual to f)

Thank you! :)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Countdown mu-calculus
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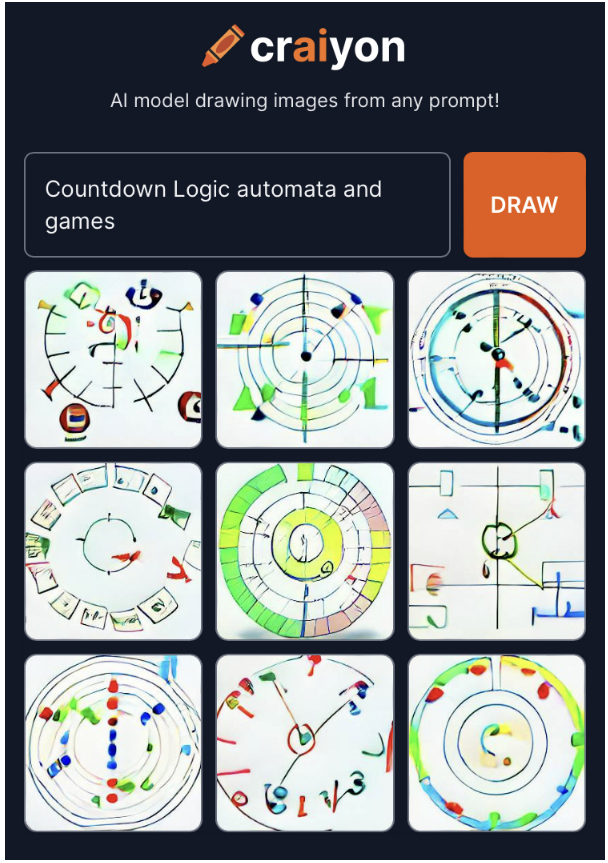
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