Deformation Theory and Moduli Spaces
IV series of exercises, for October 30

Exercise 1. Exercise 5 from series II. (The one about the flag variety, only whole exercise accepted.)

Exercise 2. Let A be a discrete valuation ring and let t be a local parameter. Let M be an A-module. Prove that the following are equivalent

(a) the A-module M is flat,

(b) the multiplication by t on M is injective.

Exercise 3. Let $A = \mathbb{k}[x, y]$ and $\mathfrak{m} = (x, y) \subseteq A$. Prove that $\mathfrak{m}$ is not flat.

Exercise 4. Let $A = \mathbb{k}[x]/(x^n)$ for $n \geq 2$ and let B be a local noetherian A-algebra. Let M be a finitely generated B-module. Prove that the following are equivalent

(a) M is a flat A-module,

(b) $\ker(\mu_x \colon M \to M) = x^{n-1}M$, where μ_x denotes the multiplication by x.

For every n give an example of M which is not flat over A, but $M/x^{n-1}M$ is flat over $\mathbb{k}[x]/(x^{n-1})$.

Exercise 5 (On reduced schemes, locally free is easy to check)**.** Let X be a scheme and $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. The *rank of $\mathcal{F}$ at a point $x \in X$* is the number $\dim_{\kappa(x)} \mathcal{F}|_x$.

(a) Show that if $\mathcal{F}$ is locally free $\mathcal{O}_X$-module of rank d, then the rank of $\mathcal{F}$ at every point $x \in X$ is indeed d.

(b) Assume now (and below) that X is reduced and that the rank of $\mathcal{F}$ is constant, equal to some number d.

(c) Suppose that $X = \operatorname{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ for an A-module M. Let $x \in X$ be a closed point of X. Prove that there is an element $f \in A$, such that $x \in (f \neq 0) \subseteq |\operatorname{Spec}(A)|$ and a surjection $A_f^{\oplus d} \to M_f$. *Hint: mimic the proof of local freeness from the lecture.*

(d) Let $\mathfrak{p}$ be a minimal prime of A. Prove that $A_{\mathfrak{p}}$ is a field, actually $A_{\mathfrak{p}} \simeq \kappa(\mathfrak{p})$. Let $(\mathfrak{p}_i)_{i \in I}$ be all minimal primes of A. Prove that the natural map $A \hookrightarrow \prod_i \kappa(\mathfrak{p}_i)$ is injective.

(e) (most important conclusion!) Localise $A_f^{\oplus d} \to M_f$ at minimal primes and conclude that it is injective, hence an isomorphism. Conclude that under the assumptions of (b) the sheaf $\mathcal{F}$ is locally free of rank d.

(f) Give an example of nonreduced X and a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ of constant rank d which is not locally free.