Deformation Theory and Moduli Spaces
III series of exercises, for October 23

Zadanie 1 (flag/nested Hilbert scheme of points)**.** Let S be a $\Bbbk$-algebra (throughout the course, the symbol $\Bbbk$ is reserved for a field) and $X = \mathrm{Spec}(S)$. Fix positive numbers $d_1 \geq d_2 \geq \ldots \geq d_s$. Consider the closed subscheme

$$\begin{aligned}\mathrm{Hilb}^{d_1,d_2,\ldots,d_s}(X) := \mathrm{Flag}^{d_1,\ldots,d_s}(S) \cap (\mathrm{Hilb}_{d_1}(X) \times \mathrm{Hilb}_{d_2}(X) \times \ldots \times \mathrm{Hilb}_{d_s}(X)) \\ \subseteq \mathrm{Gr}(d_1, S) \times \mathrm{Gr}(d_2, S) \times \ldots \times \mathrm{Gr}(d_s, S).\end{aligned}$$

Describe the $\Bbbk$-points of this scheme.

Zadanie 2. (a) Let $f\colon X \to Y$ be a morphism of schemes over some base scheme B. Prove that the following is a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id}_X, f)} & X \times_B Y \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow{\ \Delta\ } & Y \times_B Y \end{array}$$

Conclude that if $Y \to B$ is separated, then $\mathrm{graph}(f)\colon X \hookrightarrow X \times Y$ is a closed embedding, so indeed we obtain that a graph gives a closed subscheme.

(b) Let Y be a line with doubled origin, *[4.4.5, Vakil]*, let $i\colon \mathbb{A}^1 \hookrightarrow Y$ be one of the two canonical open embeddings. Consider $\mathrm{graph}(i)\colon \mathbb{A}^1 \to \mathbb{A}^1 \times Y$. Is it a closed embedding?

Zadanie 3 (The Hilbert scheme of a single point)**.** The aim of this exercise is to check that if $X \to \mathrm{Spec}(\Bbbk)$ is affine or quasi-projective, then $\mathrm{Hilb}_1(X)$ is isomorphic to X itself.

(a) Verify the claim on $\Bbbk$-points.

(b) For $f\colon S \to X$, show that $\mathrm{graph}(f) \subseteq S \times X$ is an element of $\mathrm{Hilb}_1(X)(S)$.

(c) For an element of $\mathrm{Hilb}_1(X)(S)$ which corresponds to $\mathcal{Z} \subseteq X \times S$, show that $\mathcal{Z} \to S$ is an isomorphism. Conclude that $\mathcal{Z}$ is a graph of a map $S \to X$.

(d) Prove the claim.

Zadanie 4. Exercise 5 from series II. (The one about the flag variety, only whole exercise accepted.)

Zadanie 5 ($\star$, Plücker embedding)**.** Let V be a vector space over a field $\Bbbk$ and fix $k \geq 1$.

(a) Show that $\mathrm{Gr}(1, V)$ and $\mathbb{P}(V)$ are isomorphic.[1]

(b) Recall the definition of $\Lambda^k(-)$ for a locally free sheaf and prove that $\Lambda^k(\mathcal{O}_S \otimes_{\Bbbk} V) \simeq \mathcal{O}_S \otimes_{\Bbbk} \Lambda^k V$.

(c) Consider the functor Plück: $\mathrm{Gr}(k, V) \to \mathbb{P}(\Lambda^k V)$ defined on points by

$$\text{Plück}(\mathcal{O}_S \otimes_{\Bbbk} V \twoheadrightarrow E) = (\mathcal{O}_S \otimes_{\Bbbk} \Lambda^k V \twoheadrightarrow \Lambda^k E).$$

Show that is gives a closed embedding of $\mathrm{Gr}(k, V)$ into $\mathbb{P}(\Lambda^k V)$. Conclude that if V is finite-dimensional, then $\mathrm{Gr}(k, V)$ is projective.

[1] Depending on your conventions, you might like to change V to V^* somewhere.