Deformation Theory and Moduli Spaces
II series of exercises, for October 16

Zadanie 1. (a) Let $F\colon \mathbf{Ring} \to \mathbf{Set}$ be given by $F(A) = A$. In the previous series, we showed that F is represented by $\mathbb{A}^1$. Note that A is a ring. Thus, we can promote F to the functor $F\colon \mathbf{Ring} \to \mathbf{Ring}$ given by $F(A) = A$. Show that this implies that $\mathbb{A}^1$ is a ring object in the category of affine schemes. Describe addition and multiplication in $\mathbb{A}^1$ as morphisms of schemes.

(b) Let $F\colon \mathbf{Ring} \to \mathbf{Set}$ be given by $F(A) = A^\times$, as in the previous series. The set $A^\times$ is an abelian group, so F can be promoted to $F\colon \mathbf{Ring} \to \mathbf{Ab}$. Show that this gives an abelian group structure on $\operatorname{Spec}(\mathbb{Z}[t^{\pm 1}])$ and describe it. We denote the resulting group by $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{G}_{\mathrm{m},\mathbb{Z}}$ and call it a one-dimensional torus over $\mathbb{Z}$.

Zadanie 2. Let V be a finite-dimensional vector space over $\Bbbk$.

(a) Consider the functor $\mathrm{GL}(V)\colon \mathbf{Sch}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Gr}$ given by

$$\mathrm{GL}(V)(S) := \{f : \mathcal{O}_S \otimes_{\Bbbk} V \to \mathcal{O}_S \otimes_{\Bbbk} V \mid \text{isomorphism of } \mathcal{O}_S\text{-modules}\}.$$

Show that it is represented by a group object $\mathrm{GL}(V)$ which is an affine scheme.

(b) For each positive integer k, construct an action of $\mathrm{GL}(V)$ on $\mathrm{Gr}(k, V)$ such that the action on the $\Bbbk$-points is transitive. *Hint: it's not worth writing out the morphism of schemes, that's tedious. Functors are nicer. First, check what should happen on $\Bbbk$-points. Then generalize to all S.*

Zadanie 3. Let k be a positive integer. We define $\mathrm{Gr}^{\mathrm{naive}}(k, V)\colon \mathbf{Sch}_{\Bbbk}^{\mathrm{op}} \to \mathbf{Set}$ on objects as

$$\mathrm{Gr}^{\mathrm{naive}}(k, V)(S) = \{K \subseteq \mathcal{O}_S \otimes_{\Bbbk} V \mid K \text{ locally free } \mathcal{O}_S\text{-module of rank } k\}.$$

Show that $\mathrm{Gr}^{\mathrm{naive}}(k, V)$ with the pullback *is not* a well-defined functor. *Hint: it's enough to look at $S = \mathbb{A}^1$ and $S = \mathrm{pt}$.*

Zadanie 4. Let X be a scheme and let $\varphi\colon E_1 \to E_2$ be a homomorphism of locally free $\mathcal{O}_X$-modules. We define the functor of *zeros of φ*, denoted $Z(\varphi)\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$, by

$$Z(\varphi)(S) = \{f\colon S \to X \mid f^*\varphi\colon f^*E_1 \to f^*E_2 \text{ is the zero map}\}.$$

(a) Show that if E_1, E_2 are *free* $\mathcal{O}_X$-modules, then $Z(\varphi)$ is represented by some closed subscheme of X. *Hint: in this case φ is a matrix, possibly infinite.*

(b) Let U_i be an open cover of the scheme X. Show that every $Z_i := Z(\varphi) \times_X U_i$ is an open subfunctor of $Z(\varphi)$. Show that Z_i cover Z.

(c) Consider an open cover of X that trivializes E_1, E_2 and use the theorem from the lecture to conclude that $Z(\varphi)$ is represented by a closed subscheme of X.

(d) Let $X = \mathrm{Spec}(A)$, $\mathfrak{m} \in |X|$ be a closed point. Assume that $|X|$ is connected and not a singleton[1]. Let $E_1 = \widetilde{A}$, $E_2 = \widetilde{A/\mathfrak{m}}$, thus E_2 is not locally free. Show that $Z(E_1 \twoheadrightarrow E_2)$ is not represented by a closed subscheme of X.

An important special case: if $E_1 = \mathcal{O}_X$, then φ corresponds to a section of E_2. One can reduce to this special case by considering φ as an element of $\mathrm{Hom}_{\mathcal{O}_X}(E_1, E_2)$.

Zadanie 5 (flag varieties, $\star$)**.** Let V be a $\Bbbk$-linear space. Fix positive integers s and $k_1 \geq k_2 \geq \ldots \geq k_s$. The (partial) flag variety $\mathrm{Flag}^{k_1,\ldots,k_s}(V)$ is defined by the functor

$$\mathrm{Flag}^{k_1,\ldots,k_s}(V)(S) = \{\mathcal{O}_S \otimes_{\Bbbk} V \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow E_3 \twoheadrightarrow \ldots \twoheadrightarrow E_s \mid E_i \text{ locally free of rank } k_i\}\,.$$

(a) Find a monomorphism $\mathrm{Flag}^{k_1,\ldots,k_s} \to \mathrm{Gr}(k_1, V) \times \mathrm{Gr}(k_2, V) \times \ldots \times \mathrm{Gr}(k_s, V)$.

(b) Use Exercise 4 to deduce that this morphism correspond to a closed embedding, so $\mathrm{Flag}^{k_1,\ldots k_s}(V)$ is representable.

[1] Throughout the lecture, for a scheme X the symbol $|X|$ means the underlying topological space.