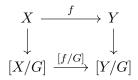
Deformation Theory and Moduli Spaces IX series of exercises, for December 11

To give you some comfort: this is the last series on stacks. We will go back to old good schemes.

Exercise 1. Let *G* be an algebraic group over \Bbbk and let $f : X \to Y$ be a *G*-equivariant map. The map f induces a map $[f/G] : [X/G] \to [Y/G]$ that on objects acts as follows:

$P \xrightarrow{\rho} X$		$P \xrightarrow{f \circ \rho} Y$
$\downarrow \pi$	is mapped to	$\downarrow \pi$
S		S

Let $X \to [X/G]$ be defined, as usually, by the trivial bundle with map $\rho: G \times X \to X$ which is the action map of G on X. Let $Y \to [Y/G]$ be defined analogously. Prove that the resulting diagram



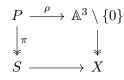
is commutative and cartesian. In the special case $Y = \text{Spec}(\mathbb{k})$, we obtain the diagram from the last lecture.

Exercise 2 (weighted projective spaces and stacks, same as in series VIII). Let $T = \mathbb{k}[x_0, x_1, x_2]$ be graded by $\deg(x_0) = 1$, $\deg(x_1) = 1$, $\deg(x_2) = 2$. This induces an action of \mathbb{G}_m on \mathbb{A}^3 . Let $X = \operatorname{Proj} T$ and $\mathcal{X} = [\mathbb{A}^3 \setminus \{0\}/\mathbb{G}_m]$. The latter is called the *weighted projective stack*.

(a) Let $c: \operatorname{Spec}(\Bbbk) \to \mathcal{X}$ be given by the orbit of $(0,0,1) \in \mathbb{A}^3 \setminus \{0\}$. Compute $\operatorname{Aut}(c)$, that is, the group scheme representing

$$\operatorname{Spec}(\Bbbk) \times_{\mathcal{X}} \operatorname{Spec}(\Bbbk).$$

(b) For every $S \to \mathcal{X}$, let $\pi \colon P \to S$ be the corresponding bundle with a \mathbb{G}_m -equivariant $\rho \colon P \to X$. Prove that there is a unique map $S \to X$ completing the diagram



Use it to construct a morphism $\mathcal{X} \to X$. *Hint: during the exercises, we observed that* $P \to S$ *is the categorical quotient, that is, any G-invariant map from P factors through S.*

(c) \star . Prove that the map $\mathcal{X} \to X$ is an isomorphism over the open subset

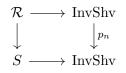
$$X \setminus \{[0:0:1]\} = (x_0 \neq 0) \cup (x_1 \neq 0) \subseteq X.$$

Hint: everything is local on X.

The stack \mathcal{X} is smooth by definition; we can think of \mathcal{X} as a stacky resolution of singularities of X.

Exercise 3. Recall the stack InvShv \rightarrow Sch_k, isomorphic to $B\mathbb{G}_m$, defined on the last lecture: the objects of InvShv are pairs (S, L), where L is an invertible sheaf on a k-scheme S. The morphisms $(S', L') \rightarrow (S, L)$ are pairs (f, f^{\flat}) , where $f: S' \rightarrow S$ and $f^{\flat}: L' \rightarrow f^*L$ is an isomorphism of invertible sheaves.

- (a) For an integer *n*, define a morphism p_n : InvShv \rightarrow InvShv which on objects maps (S, L) to (S, L^n) .¹
- (b) Fix a line bundle L on a \Bbbk -scheme S and consider the pullback



Prove that the objects of \mathcal{R} correspond to line bundles whole *n*-th power is the pullback of *L*. (Part of the exercise is to make this precise.)

Exercise 4 (an exercise in unravelling definitions). Let \Bbbk be an algebraically closed field. Let $\mathcal{M}_{g,n}$ be the *stack of n-pointed genus g curves*. This means that the objects of $\mathcal{M}_{g,n}$ are pairs $(\pi, (s_1, \ldots, s_n))$, where

- *π*: *C* → *S* is a smooth proper morphism such that for every point *s* ∈ *S*, the fibre *π*⁻¹(*s*) is a curve (smooth and proper by the previous assumptions) of genus *g*,
- $s_1, \ldots, s_n \colon S \to C$ are sections of π such that $s_i(S) \cap s_j(S) = \emptyset$ for $i \neq j$.

The morphisms from $(\pi', (s'_1, \ldots, s'_n))$ to $(\pi, (s_1, \ldots, s_n))$ are pairs (f, f^{\flat}) , where $f : S' \to S$ is a morphism of k-schemes, $f^{\flat} : C' \to C \times_S S'$ is an isomorphism of S'-schemes which, for every *i*, identifies the section s'_i with $s_i \times id_{S'} : S' = S \times_S S' \to C \times_S S'$.

- (a) Let $\mathbb{P}^1 \to \operatorname{Spec}(\Bbbk)$ be a curve and $c: \operatorname{Spec}(\Bbbk) \to \mathcal{M}_{0,0}$ be corresponding morphism. Compute the automorphism group $\operatorname{Aut}(c)$. *Hint: unwind the definitions and see that you most likely know the answer already from Vakil's book.*
- (b) Fix a k-point on \mathbb{P}^1 and the corresponding morphism $c_1 \colon \operatorname{Spec}(\Bbbk) \to \mathcal{M}_{0,1}$. Compute $\operatorname{Aut}(c_1)$.
- (c) Do the same for $\mathcal{M}_{0,2}$ and two fixed k-points on \mathbb{P}^1 .
- (d) Prove that any two objects in $\mathcal{M}_{0,3}(\Bbbk)$ are isomorphic in a unique way.
- (e) * Prove that $\mathcal{M}_{0,3}$ is isomorphic to $\operatorname{Spec}(\Bbbk)$. *Hint: you'll need to know that* $C \to S$ *is isomorphic to* $\mathbb{P}(E) \to S$ *for a vector bundle* E. *See Hartshorne's book.*

 $^{{}^{1}}p_{n}$ stands for "*n*-th power".