Deformation Theory and Moduli Spaces
IX series of exercises, for December 11

To give you some comfort: this is the last series on stacks. We will go back to old good schemes.

Exercise 1. Let G be an algebraic group over $\Bbbk$ and let $f\colon X \to Y$ be a G-equivariant map. The map f induces a map $[f/G]\colon [X/G] \to [Y/G]$ that on objects acts as follows:

$$\begin{array}{ccc} P & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle \pi} & & \\ S & & \end{array} \qquad \text{is mapped to} \qquad \begin{array}{ccc} P & \xrightarrow{f\circ\rho} & Y \\ \downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

Let $X \to [X/G]$ be defined, as usually, by the trivial bundle with map $\rho\colon G \times X \to X$ which is the action map of G on X. Let $Y \to [Y/G]$ be defined analogously. Prove that the resulting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ [X/G] & \xrightarrow{[f/G]} & [Y/G] \end{array}$$

is commutative and cartesian. *In the special case* $Y = \operatorname{Spec}(\Bbbk)$, *we obtain the diagram from the last lecture.*

Exercise 2 (weighted projective spaces and stacks, same as in series VIII)**.** Let $T = \Bbbk[x_0, x_1, x_2]$ be graded by $\deg(x_0) = 1$, $\deg(x_1) = 1$, $\deg(x_2) = 2$. This induces an action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^3$. Let $X = \operatorname{Proj} T$ and $\mathcal{X} = [\mathbb{A}^3 \setminus \{0\}/\mathbb{G}_{\mathrm{m}}]$. The latter is called the *weighted projective stack*.

(a) Let $c\colon \operatorname{Spec}(\Bbbk) \to \mathcal{X}$ be given by the orbit of $(0, 0, 1) \in \mathbb{A}^3 \setminus \{0\}$. Compute $\operatorname{Aut}(c)$, that is, the group scheme representing
$$\operatorname{Spec}(\Bbbk) \times_{\mathcal{X}} \operatorname{Spec}(\Bbbk).$$

(b) For every $S \to \mathcal{X}$, let $\pi\colon P \to S$ be the corresponding bundle with a $\mathbb{G}_{\mathrm{m}}$-equivariant $\rho\colon P \to X$. Prove that there is a unique map $S \to X$ completing the diagram

$$\begin{array}{ccc} P & \xrightarrow{\rho} & \mathbb{A}^3 \setminus \{0\} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ S & \longrightarrow & X \end{array}$$

Use it to construct a morphism $\mathcal{X} \to X$. *Hint: during the exercises, we observed that $P \to S$ is the categorical quotient, that is, any G-invariant map from P factors through S.*

(c) $\star$. Prove that the map $\mathcal{X} \to X$ is an isomorphism over the open subset

$$X \setminus \{[0 : 0 : 1]\} = (x_0 \neq 0) \cup (x_1 \neq 0) \subseteq X.$$

Hint: everything is local on X.

The stack $\mathcal{X}$ is smooth by definition; we can think of $\mathcal{X}$ as a stacky resolution of singularities of X.

Exercise 3. Recall the stack $\operatorname{InvShv} \to \mathbf{Sch}_{\Bbbk}$, isomorphic to $B\mathbb{G}_{\mathrm{m}}$, defined on the last lecture: the objects of InvShv are pairs (S, L), where L is an invertible sheaf on a $\Bbbk$-scheme S. The morphisms $(S', L') \to (S, L)$ are pairs $(f, f^\flat)$, where $f\colon S' \to S$ and $f^\flat\colon L' \to f^*L$ is an isomorphism of invertible sheaves.

(a) For an integer n, define a morphism $p_n\colon \mathrm{InvShv} \to \mathrm{InvShv}$ which on objects maps (S, L) to (S, L^n).[1]

(b) Fix a line bundle L on a $\Bbbk$-scheme S and consider the pullback

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathrm{InvShv} \\ \big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ S & \longrightarrow & \mathrm{InvShv} \end{array}$$

Prove that the objects of $\mathcal{R}$ correspond to line bundles whole n-th power is the pullback of L. (Part of the exercise is to make this precise.)

Exercise 4 (an exercise in unravelling definitions)**.** Let $\Bbbk$ be an algebraically closed field. Let $\mathcal{M}_{g,n}$ be the *stack of n-pointed genus g curves*. This means that the objects of $\mathcal{M}_{g,n}$ are pairs $(\pi, (s_1, \ldots, s_n))$, where

- $\pi\colon C \to S$ is a smooth proper morphism such that for every point $s \in S$, the fibre $\pi^{-1}(s)$ is a curve (smooth and proper by the previous assumptions) of genus g,
- $s_1, \ldots, s_n\colon S \to C$ are sections of π such that $s_i(S) \cap s_j(S) = \emptyset$ for $i \neq j$.

The morphisms from $(\pi', (s'_1, \ldots, s'_n))$ to $(\pi, (s_1, \ldots, s_n))$ are pairs $(f, f^\flat)$, where $f\colon S' \to S$ is a morphism of $\Bbbk$-schemes, $f^\flat\colon C' \to C \times_S S'$ is an isomorphism of S'-schemes which, for every i, identifies the section s'_i with $s_i \times \mathrm{id}_{S'}\colon S' = S \times_S S' \to C \times_S S'$.

(a) Let $\mathbb{P}^1 \to \mathrm{Spec}(\Bbbk)$ be a curve and $c\colon \mathrm{Spec}(\Bbbk) \to \mathcal{M}_{0,0}$ be corresponding morphism. Compute the automorphism group $\mathrm{Aut}(c)$. *Hint: unwind the definitions and see that you most likely know the answer already from Vakil's book.*

(b) Fix a $\Bbbk$-point on $\mathbb{P}^1$ and the corresponding morphism $c_1\colon \mathrm{Spec}(\Bbbk) \to \mathcal{M}_{0,1}$. Compute $\mathrm{Aut}(c_1)$.

(c) Do the same for $\mathcal{M}_{0,2}$ and two fixed $\Bbbk$-points on $\mathbb{P}^1$.

(d) Prove that any two objects in $\mathcal{M}_{0,3}(\Bbbk)$ are isomorphic in a unique way.

(e) $\star$ Prove that $\mathcal{M}_{0,3}$ is isomorphic to $\mathrm{Spec}(\Bbbk)$. *Hint: you'll need to know that $C \to S$ is isomorphic to $\mathbb{P}(E) \to S$ for a vector bundle E. See Hartshorne's book.*

[1] p_n stands for "n-th power".