Deformation Theory and Moduli Spaces
VIII series of exercises, for November 27

Exercise 1. Let G be an algebraic group over $\Bbbk$. Let $\pi\colon P \to S$ be a principal G-bundle over a $\Bbbk$-scheme S and $S \to BG$ be the corresponding morphism[1]. Let $\star = \operatorname{Spec}(\Bbbk)$.

(a) Prove that

$$\begin{array}{ccc} P & \longrightarrow & \star \\ {\scriptstyle\pi}\big\downarrow & \lrcorner & \big\downarrow \\ S & \longrightarrow & BG \end{array}$$

is a pullback diagram in the category of fibered cats over $\mathbf{Sch}_\Bbbk$.

(b) Let X be a $\Bbbk$-scheme with G-action and suppose that we have a G-equivariant map $\rho\colon P \to X$ of $\Bbbk$-schemes. Let again $S \to [X/G]$ be the corresponding morphism. Prove that

$$\begin{array}{ccc} P & \xrightarrow{\ \rho\ } & X \\ {\scriptstyle\pi}\big\downarrow & \lrcorner & \big\downarrow \\ S & \longrightarrow & [X/G] \end{array}$$

is a pullback diagram in the category of fibered cats over $\mathbf{Sch}_\Bbbk$.

(c) Conclude that $X \to [X/G]$ is a smooth surjective map. *If you want, you can take for granted that $[X/G]$ has diagonal representable by schemes.*

Exercise 2 (Some motivation for the name $[X/G]$)**.** Let $\Bbbk$ be an algebraically closed field. Let X be a $\Bbbk$-scheme with an action of an algebraic group $G \to \operatorname{Spec}(\Bbbk)$.

(a) Prove that any G-bundle over $\operatorname{Spec}(\Bbbk)$ is isomorphic to a trivial bundle. *Hint: $P(\Bbbk) \neq \emptyset$.*

(b) Prove that the groupoid $[X/G](\operatorname{Spec}(\Bbbk))$[2] is equivalent (as a category) to the set of $G(\Bbbk)$-orbits on $X(\Bbbk)$.

Exercise 3. Let $\mathbb{P}^n$ be the projective space and $\pi\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the usual map. Let $\mathbb{G}_\mathrm{m}$ act on $\mathbb{A}^{n+1}$ by scalar multiplication.

(a) Prove that π is a $\mathbb{G}_\mathrm{m}$-bundle, hence it yields a map $\mathbb{P}^n \to [\mathbb{A}^{n+1} \setminus \{0\}/\mathbb{G}_\mathrm{m}]$.

(b) $\star$ Prove that $[\mathbb{A}^{n+1} \setminus \{0\}/\mathbb{G}_\mathrm{m}] \simeq \mathbb{P}^n$.

(c) Prove that $[\mathbb{A}^{n+1} \setminus \{0\}/\mathbb{G}_\mathrm{m}] \to [\mathbb{A}^{n+1}/\mathbb{G}_\mathrm{m}]$ is an open immersion[3]. Hence $\mathbb{P}^n$ is open(!) inside $[\mathbb{A}^{n+1}/\mathbb{G}_\mathrm{m}]$.

Exercise 4 (weighted projective spaces and stacks)**.** Let $T = \Bbbk[x_0, \dots, x_n]$ be a polynomial ring graded by $\deg(x_i) = a_i \in \mathbb{Z}_+$. The scheme $X = \operatorname{Proj} T$ is called the *weighted projective space*.

(a) Suppose that $a_0 = 1$. Prove that $(x_0 \neq 0) \subseteq X$ is isomorphic to $\mathbb{A}^n$.

(b) Consider $(a_0, a_1, a_2) = (1, 1, 2)$. Prove that the point $[0 : 0 : 1] \in X$ is the unique singular point of X. Prove that the natural map $\mathbb{A}^3 \setminus \{0\} \to X$ is not a $\mathbb{G}_\mathrm{m}$-bundle. *Hint: fibre over $[0 : 0 : 1]$.*

[1]We view the scheme S as the fibered category and/or use Yoneda lemma as in the lecture

[2]As defined on the lecture: it is called the fiber category.

[3]Which means that the pullback by any map $S \to [\mathbb{A}^{n+1}/\mathbb{G}_\mathrm{m}]$ from a scheme yields an open immersion of schemes.

(c) Let $\mathcal{X} = [\mathbb{A}^3 \setminus \{0\}/\mathbb{G}_{\mathrm{m}}]$, where the action of $\mathbb{G}_{\mathrm{m}}$ is $t \cdot (x_0, x_1, x_2) = (tx_0, tx_1, t^2 x_2)$. Let $\mathrm{Spec}(\Bbbk) \to \mathcal{X}$ be given by the orbit map $\mathbb{G}_{\mathrm{m}} \to \mathbb{A}^3 \setminus \{0\}$ that sends $1_{\mathbb{G}_{\mathrm{m}}}$ to $(0, 0, 1)$. Compute the scheme representing

$$\mathrm{Spec}(\Bbbk) \times_{\mathcal{X}} \mathrm{Spec}(\Bbbk).$$

(d) $\star$. For every $S \to [\mathbb{A}^3 \setminus \{0\}/\mathbb{G}_{\mathrm{m}}]$, let $\pi \colon P \to S$ be the corresponding bundle with a $\mathbb{G}_{\mathrm{m}}$-equivariant $\rho \colon P \to X$. Prove that there is a unique map $S \to X$ completing the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \rho\ } & \mathbb{A}^3 \setminus \{0\} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ S & \longrightarrow & X \end{array}$$

Use it to construct a morphism $\mathcal{X} \to X$.

(e) $\star$. Prove that the map $\mathcal{X} \to X$ is an isomorphism over the open subset

$$X \setminus \{[0 : 0 : 1]\} = (x_0 \neq 0) \cup (x_1 \neq 0) \subseteq X.$$

Hint: everything is local on X. Repeat the argument from Ex 3(b)