Deformation Theory and Moduli Spaces

VII series of exercises, for November 20

Exercise 1. Let $p\colon F \to C$ be a category fibered in groupoids. Prove that every morphism $\phi\colon u \to v$ can be factored as $\phi = \psi \circ \tau$, where τ is cartesian with $p(\tau) = p(\phi)$ and ψ is an isomorphism in $F(p(u))$. Conclude that every morphism is cartesian.

Exercise 2. Let X be a scheme. Consider the category h_X[1] whose objects are morphisms $S \to X$ and

$$\mathrm{Mor}(f'\colon S' \to X,\, f\colon S \to X) = \left\{g\colon S' \to S \mid f' = f \circ g\right\}.$$

Let $p\colon h_X \to \mathbf{Sch}$ be the functor that sends $S \to X$ to S.

(a) Prove that $p\colon h_X \to \mathbf{Sch}$ is fibered in setoids.

(b) Prove that $\mathrm{Mor}_{\mathbf{FibCats}}(h_X, h_Y) \simeq \mathrm{Mor}_{\mathbf{Sch}}(X, Y)$.

Exercise 3 (Some non-bundles)**.** Let G be an algebraic group over $\Bbbk$.

(a) Let $S = \mathrm{Spec}(\Bbbk[t])$, $S' = \mathrm{Spec}(\Bbbk[t^{\pm 1}]) \hookrightarrow S$ and $P = G \times_{\Bbbk} S'$. Prove that $G \times_{\Bbbk} P \simeq P \times_S P$ and $P \to S$ is flat. Yet P is not a principal G-bundle.

(b) Let $S = \mathbb{A}^1$ and $S' = (\mathbb{A}^1 \setminus \{0\}) \sqcup \mathrm{Spec}(\Bbbk)$. Consider $S' \to S$ that sends $\mathrm{Spec}(\Bbbk)$ isomorphically to zero and $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is the natural inclusion. Let $P = G \times_{\Bbbk} S'$. Prove that $G \times_{\Bbbk} P \simeq P \times_S P$ and $P \to S$ is surjective. Yet P is not a principal G-bundle.

Exercise 4 (Examples of non-Zariski trivializable bundles)**.** (a) Let $n > 0$ be a number. Let $\Bbbk$ be algebraically closed and of characteristic not dividing n. Consider the pullback

$$\begin{array}{ccc}
\mu_n & \longrightarrow & \mathbb{G}_{\mathrm{m}} \\
\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle z \mapsto z^n} \\
\mathrm{Spec}(\Bbbk) & \xrightarrow{1_{\mathbb{G}_{\mathrm{m}}}} & \mathbb{G}_{\mathrm{m}}
\end{array}$$

Prove that μ_n is a reduced group scheme such that the map $\mu_n \to \mathbb{G}_{\mathrm{m}}$ is an inclusion of group schemes. Prove that the map $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is a μ_n-bundle which is not Zariski-trivializable.

(b) Let $q = x_1^2 + x_2^2$ and $q' = x_1^2 - x_2^2$. Consider the affine $\mathbb{R}$-schemes given by the functors $P\colon \mathbf{Alg}_{\mathbb{R}} \to \mathbf{Set}$, $\mathrm{O}_q\colon \mathbf{Alg}_{\mathbb{R}} \to \mathbf{Set}$ defined by

$$\begin{aligned}
\mathrm{O}_q(A) &= \{M \in \mathrm{GL}_2(A) \mid M \cdot q = q\}, \\
P(A) &= \left\{M \in \mathrm{GL}_2(A) \mid M \cdot q' = q\right\}.
\end{aligned}$$

Prove that O_q is a group scheme, that we have an action $\mathrm{O}_q \times_{\mathbb{R}} P \to P$ and that $P(\mathbb{R}) = \emptyset$. Conclude that P is not a Zariski-trivializable O_q-bundle.

(c) In the notation of (b), prove that P is an O_q bundle. *Hint:* $P_{\mathbb{C}} \simeq (\mathrm{O}_q)_{\mathbb{C}}$.

[1] Note that h_X was defined differently (wrongly) on the last lecture. Apologies for this!

Exercise 5. Let $i_1 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be given by $[u : v] \mapsto [u^3 : u^2v : uv^2 : v^3]$. Consider the $\mathbb{G}_{\mathrm{m}}$ action on $\mathbb{P}^3$ which rescales the third coordinate: $t \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : tx_2 : x_3]$. It yields the orbit map $\mathbb{G}_{\mathrm{m}} \ni t \to t[i_1(\mathbb{P}^1)] \in \mathrm{Hilb}(\mathbb{P}^3)$ which extends to $f \colon \mathbb{P}^1 \to \mathrm{Hilb}(\mathbb{P}^3)$ by Exercise V.3. Let $C_0 = f(0)$ be the limit of $i_1(\mathbb{P}^1)$ as t goes to zero.

(a) Prove that $x_2^2,\ x_0^2x_3 - x_1^3 \in I_{C_0}$. Conclude that $|C_0| = V(x_0^2x_3 - x_1^3,\ x_2)$. *Hint: you can and should use the method of computing C_0 shown by Prajwal last time.*

(b) Compute the Hilbert polynomial of $V(x_0^2x_3 - x_1^3,\ x_2)$ and conclude that $C_0 \neq V(x_0^2x_3 - x_1^3,\ x_2)$.

(c) Why (a) and (b) do not contradict each other?

(d) $\star$ Compute I_{C_0}.