Deformation Theory and Moduli Spaces
VI series of exercises, for November 13

Exercise 1. Let $\Bbbk$ be a field, let $p = (0, 0, \ldots, 0) \in \Bbbk^n$ be the origin and let $Z_n = V(\mathfrak{m}_p^2) \subseteq \mathbb{A}^n_{\Bbbk}$.

(a) Find the degree of Z_n and the dimension of the tangent space $T_{[Z_n]} \operatorname{Hilb}(\mathbb{A}^n_{\Bbbk})$.

(b) Prove that $\mathcal{Z}_n$ given by the ideal

$$\frac{\Bbbk[x_1, \ldots, x_n][t]}{(x_1(x_1 - t), x_1x_2, \ldots, x_1x_n, x_2(x_2 - t), x_2x_3, \ldots, x_3(x_3 - t), \ldots, x_{n-1}x_n)}$$

is finite and flat over $\Bbbk[t]$ and proves that that $[Z]$ is in the closure of the locus of tuples of points.

(c) Prove that the Hilbert scheme $\operatorname{Hilb}_4(\mathbb{A}^3)$ is singular. *Hint: You can use the fact that it has a component of dimension* $3 \cdot 4$*, see lecture.*

Exercise 2 (Harthsorne's *distraction method*)**.** Let $\Bbbk$ have characteristic zero[1]. Let $I \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal. Suppose that $d = \dim_{\Bbbk} S/I$ is finite. For a monomial $x_1^{a_1} \ldots x_n^{a_n} \in S$ its *distration* is the product

$$x_1(x_1-t)(x_1-2t) \ldots (x_1-(a_1-1)t) \cdot x_2(x_2-t) \ldots (x_2-(a_2-1)t) \cdot \ldots \cdot x_n(x_n-t) \ldots (x_n-(a_n-1)t) \in S[t].$$

(a) Let $I' \subseteq S[t]$ be the ideal generated by the distractions of monomials in I. Prove that for a nonzero $\lambda \in \Bbbk$, the fibre $\operatorname{Spec}(S[t]/I')|_{t=\lambda}$ has exactly d points. Prove that every fibre over a nonzero λ is isomorphic to the fibre over $\lambda = 1$.

(b) Prove that $S[t]/I'$ is a finitely generated $\Bbbk[t]$-module. Prove that it is locally free of rank d. *Hint: semicontinuity and the previous point give one way to prove flatness without any computations.* Conclude that S/I is in the closure of the locus of tuples of points.

Exercise 3 (Iarrobino-Emsalem's example)**.** Let $Z \subseteq \mathbb{A}^4_{\Bbbk}$ be given by the ideal

$$I_Z = \left(x_1^2,\, x_1x_2,\, x_2^2,\, x_3^2,\, x_3x_4,\, x_4^2,\, x_1x_3 - x_2x_4\right) \subseteq \Bbbk[x_1, x_2, x_3, x_4].$$

One can compute that Z has degree 8 and that $\dim_{\Bbbk} T_{[Z]} \operatorname{Hilb}_8(\mathbb{A}^4)$ is equal to 25. Prove that $[Z]$ is not a limit of tuples of points. Conclude that $\operatorname{Hilb}_8(\mathbb{A}^4)$ is reducible.
Note: it is known that $\operatorname{Hilb}_d(\mathbb{A}^3)$ *is reducible for* $d \gg 0$*, but the exact value is unknown, this is open since '70. In contrast,* $\operatorname{Hilb}_d(\mathbb{A}^n)$ *is irreducible for every* $d \le 7$ *and every* n*.*

Exercise 4. Let $i_1 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be given by $[u : v] \mapsto [u^3 : u^2v : uv^2 : v^3]$ and $i_2 \colon \mathbb{P}^1 \to \mathbb{P}^3$ be given by $[u : v] \mapsto [u^2 : uv : v^2 : 0]$. Consider the $\mathbb{G}_{\mathrm{m}}$ action on $\mathbb{P}^3$ which rescales the last coordinate: $t \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : x_2 : tx_3]$. It yields the orbit map $\mathbb{G}_{\mathrm{m}} \ni t \to t[i_1(\mathbb{P}^1)] \in \operatorname{Hilb}(\mathbb{P}^3)$ which extends to $f \colon \mathbb{P}^1 \to \operatorname{Hilb}(\mathbb{P}^3)$ by Exercise V.3. The curves $C_0 = f(0)$, $C_\infty = f(\infty)$ are called the *limits of* $i_1(\mathbb{P}^1)$ *as* t *goes to zero or* ∞*, respectively.*

[1] Actually it is enough to have an infinite field

(a) Prove that $i_2(\mathbb{P}^1) = V(x_0x_2 - x_1^2, x_3)$. Compute the Hilbert polynomials of C_0 and of $i_2(\mathbb{P}^3)$ and conclude that these two curves lie in different connected components of the Hilbert scheme. *Hint: no need to compute I_{C_0}.*

(b) Prove that C_0 is $\mathbb{G}_{\mathrm{m}}$-stable. Conclude that, topologically, it is contained in $\mathbb{P}^2 = [* : * : * : 0]$.

(c) Prove that the homogeneous ideal I_{C_0} contains $x_0x_2 - x_1^2$. *Hint: no genuine calculations are necessary.* Conclude that $|C_0| = |V(x_0x_2 - x_1^2, x_3)|$ as sets.

(d) Why (a) and (c) do not contradict each other?

(e) $\star$ Compute I_{C_0}.