Deformation Theory and Moduli Spaces
V series of exercises, for November 6

Exercise 1 (The Hilbert scheme of two points)**.** Let $X = \mathbb{A}^n_{\Bbbk}$, where $\Bbbk$ is an algebraically closed field.

(a) Let $Z \subseteq X$ correspond to a $\Bbbk$-point $[Z] \in \mathrm{Hilb}_2(X)$. Prove that exactly one of the following situations holds:

- Z is reduced and corresponds to an (unordered) pair of $\Bbbk$-points of Z,
- Z is nonreduced and it corresponds to a tangent vector at a $\Bbbk$-point $p \in \Bbbk^n$.

Hint: lecture and/or Exercise I.2 (which is written in Polish, sorry).

(b) For every Z as above, prove that the dimension of the tangent space at $[Z] \in \mathrm{Hilb}_2(X)$ is $2n$.

(c) Prove that for every nonreduced $Z \subseteq X$ as above, there is a morphism $f\colon \mathbb{A}^1 \to \mathrm{Hilb}_2(X)$ such that $f(\lambda)$ is reduced for $0 \neq \lambda \in \Bbbk$ and $f(0) = [Z]$.

Exercise 2 (The Hilbert scheme of two points, continued)**.** Let $X = \mathbb{A}^n_{\Bbbk}$, where $\Bbbk$ is an algebraically closed field.

(a) Let $X \simeq \Delta \hookrightarrow X^2$ be the diagonal. Prove that Δ is closed. Let $U = X^2 \setminus \Delta$ be the complement. Construct a closed subscheme $Z \subseteq U \times X$ such that

$$Z(\Bbbk) = \{(x, y, x) \mid x, y \in \Bbbk^n,\ x \neq y\} \cup \{(x, y, y) \mid x, y \in \Bbbk^n,\ x \neq y\} \subseteq U(\Bbbk) \times \Bbbk^n.$$

(b) Prove that the canonical map $f\colon Z \to U$ is affine and $f_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_U$-module of rank two, so that f is finite. This yields a morphism $U \to \mathrm{Hilb}_2(X)$.

(c) Prove that the closure of the (topological) image of U is equal to whole $\mathrm{Hilb}_2(X)$. *Hint: this is a purely formal (yet nontrivial to deduce!) consequence of previous exercise and base change to avoid non-closed fields.*

(d) Conclude that $\mathrm{Hilb}_2(X)$ is irreducible and it has dimension $2n$.

(e) Deduce that $\mathrm{Hilb}_2(X)$ is smooth.

(f) Prove that the morphism $U \to \mathrm{Hilb}_2(X)$ cannot be extended to a continuous mapping defined on an open $V \subseteq X^2$ which strictly contains U.

Exercise 3 (Flat limits)**.** Let X be a projective scheme. Let C be a smooth curve, $Z \subseteq C$ be a proper closed subset and $f\colon C \setminus Z \to \mathrm{Hilb}^p(X)$. Let $\mathcal{Z} \subseteq (C \setminus Z) \times X$ be the corresponding flat family.

(a) Let $\overline{\mathcal{Z}} \subseteq C \times X$ be the closure of $\mathcal{Z}$ in $C \times X$. Prove that $\overline{\mathcal{Z}} \to C$ is flat and that the corresponding morphism $C \to \mathrm{Hilb}^p(X)$ extends f.

(b) Find an example of a *singular* curve C and of Z, X, and $\mathcal{Z}$, such that the extension as in (a) does not exist.