Deformation Theory and Moduli Spaces

XI series of exercises, for January 8

Throughout, the term *an Artinian local* $\mathbb{k}$*-algebra* $(R, \mathfrak{m}, \mathbb{k})$ means an Artinian $\mathbb{k}$-algebra R which is local with maximal ideal $\mathfrak{m}$ such that $\mathbb{k} \hookrightarrow R/\mathfrak{m}$ is bijective; in other words, the notation $(R, \mathfrak{m}, \mathbb{k})$ in particular implies that the unique point of $\operatorname{Spec}(R)$ is a $\mathbb{k}$-point. For a $\mathbb{k}$-algebra S we write S_R to denote $S \otimes_{\mathbb{k}} R$.

Exercise 1. Let $(R, \mathfrak{m}, \mathbb{k})$ be an Artinian local $\mathbb{k}$-algebra. Let $S = \mathbb{k}[x_1, \ldots, x_n]$. Let $X = \operatorname{Spec}(A)$ be an $\mathbb{k}$-algebra and let $\mathcal{X} \to \operatorname{Spec}(R)$ be such that there is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \text{flat}} \\ \operatorname{Spec}(\mathbb{k}) & \xrightarrow{\text{cl}} & \operatorname{Spec}(R) \end{array}$$

where $\operatorname{Spec}(\mathbb{k}) \to \operatorname{Spec}(R)$ is the canonical map. We say that $\mathcal{X}$ is a *deformation* of X over $\operatorname{Spec}(R)$.

(a) Suppose that $X \subseteq \operatorname{Spec}(S)$ is a closed embedding. Suppose that $\mathcal{X}$ is affine. Prove that there is a closed embedding $\mathcal{X} \subseteq \operatorname{Spec}(S_R)$. *Hint: pass to rings and choose any* $S_R \to H^0(\mathcal{O}_{\mathcal{X}})$ *that yields the given* $S \twoheadrightarrow H^0(\mathcal{O}_X)$.

(b) $\star\star$. Prove that if X is affine, then $\mathcal{X}$ is also affine. Hence, in the setting of (a), the assumption that $\mathcal{X}$ is affine was unnecessary. *Hint: this is general algebraic geometry fact. It is key that* $\operatorname{Spec}(R)$ *is Artinian.*

(c) Suppose that $\mathcal{X}$ is a closed subscheme of $\operatorname{Spec}(S_R)$, so that $\mathcal{X} \simeq \operatorname{Spec}(S_R/I)$ and $X \simeq \operatorname{Spec}(S/\bar{I})$, where $\bar{I} \subseteq S$ is the image of I. Suppose that the ideal $\bar{I}$ is generated by r elements $\bar{f}_1, \ldots, \bar{f}_r$ and let $f_1, \ldots, f_r \in I$ be any preimages. Prove that $I = (f_1, \ldots, f_r)$.

(d) Let $X = \operatorname{Spec}(\mathbb{k}[x, y]/(xy))$ and let $\mathcal{X} \to \operatorname{Spec}(R)$ be affine. Prove that

$$\mathcal{X} \simeq \operatorname{Spec}(R[x, y]/(xy - f)) \tag{1.1}$$

for some polynomial $f \in R[x, y]$. *Thus there are no unexpected deformations.*

(e) Conversely, prove that for every f, the scheme (1.1) is flat over $\operatorname{Spec}(R)$, so it is indeed a deformation of $\operatorname{Spec}(\mathbb{k}[x, y]/(xy))$. *Hint: for example, local criterion for flatness. It is key that there is only one equation.*

(f) Consider a deformation $\mathcal{X} = \operatorname{Spec}(R[x, y]/(xy - f))$. Let $f_0 \in R$ be the constant term of f. Let $\mathcal{X}' = \operatorname{Spec}(R[x, y]/(xy - f_0))$. Prove that $\mathcal{X}, \mathcal{X}'$ are isomorphic deformations. (That is, they are isomorphic as R-schemes and in such a way that the isomorphism restricts to the identity map from $X \subseteq \mathcal{X}$ to $X \subseteq \mathcal{X}'$.) *Hint: the isomorphism is very much NOT* $R[x, y]$*-linear. Use derivations as in Exercise X.1. You will need to get your hards dirty. You might like to do* $R = \mathbb{k}[\varepsilon]/\varepsilon^2$ *first.*

(g) Let $\mathcal{V} := \operatorname{Spec}(\mathbb{k}[x, y][[t]]/(xy - t)) \to \operatorname{Spec}(\mathbb{k}[[t]])$ be the morphism from last series; it is flat. Prove that every $\mathcal{X}$ in (d) is a pullback of $\mathcal{V}$, that is, there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & \operatorname{Spec}(\mathbb{k}[[t]]) \end{array}$$

and the map $\mathcal{X} \to \mathcal{V}$ restricts to the identity from $X \subseteq \mathcal{X}$ to $X \subseteq \mathcal{V}$. We say that $\mathcal{V}$ is *versal*.

Exercise 2. Let $S = \Bbbk[x_1, \ldots, x_n]$ and let $I \subseteq S$ be an ideal such that $X = \mathrm{Spec}(S/I)$ is a smooth $\Bbbk$-scheme. The aim of this exercise is to prove that $T^1_{S/I}$, as defined in Exercise X.1, is zero.

Let $\varphi \in \mathrm{Hom}_S(I, S/I)$ be a tangent vector and $\mathcal{X} := \mathrm{Spec}\left(S_{\Bbbk[\varepsilon]/\varepsilon^2}/\mathcal{I}\right)$ be the corresponding scheme, which is flat over $\mathrm{Spec}(\Bbbk[\varepsilon]/\varepsilon^2)$. By the lecture, we have an isomorphism of $\Bbbk[\varepsilon]/\varepsilon^2$-algebras $S_{\Bbbk[\varepsilon]/\varepsilon^2}/\mathcal{I} \simeq S[\varepsilon]/\varepsilon^2$. Use it to deduce that there is a derivation $d\colon S \to S/I$ such that $\varphi = d|_I$. Conclude that $T^1_{S/I}$ is zero.

Exercise 3. Let $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be given by $[u : v] \mapsto [u^3, u^2v : uv^2 : v^3]$. On the one hand, during the last lecture we observed that every deformation of $\mathbb{P}^1$ over $\Bbbk[\varepsilon]/\varepsilon^2$ is trivial. On the other hand, long ago we observed that the tangent space to $[\mathbb{P}^1 \hookrightarrow \mathbb{P}^3]$ in the Hilbert scheme is nontrivial. Explain why there is no contradiction.

Exercise 4. $\star$. Let p be a prime number.

(a) Let $X \to \mathrm{Spec}(\mathbb{F}_p)$ be a smooth affine scheme. Prove that it has a deformation $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ are the p-adic numbers. *Hint: infinitesimal lifting.*

(b) Let $C \to \mathrm{Spec}(\mathbb{F}_p)$ be a smooth projective curve. Prove that it has a deformation $\mathcal{C} \to \mathrm{Spec}(\mathbb{Z}_p)$, were $\mathbb{Z}_p$ are the p-adic numbers. *Hint: take two affine pieces and lift each of them.*

For smooth surfaces and in higher dimensions, the lifting in general does not exist.