Deformation Theory and Moduli Spaces
X series of exercises, for December 18

Now, we shift focus for the "global" picture of representable functors and stacks to the "local" picture, where we care only about morphisms to these stacks from $\mathrm{Spec}(A)$, where A is local Artinian (hence topologically a point). Also, this series is shorter to allow you to solve/struggle with all exercises.

Exercise 1. Let $I \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be an ideal.

(a) Let $d\colon S \to S/I$ be a derivation. Prove that $d|_I\colon I \to S/I$ is S-linear, so $d|_I \in \mathrm{Hom}_S(I, S/I)$.

(b) Recall (from a lecture long ago) that to $\varphi \in \mathrm{Hom}_S(I, S/I)$ we may associate a flat $\Bbbk[\varepsilon]/\varepsilon^2$-algebra which is a quotient of $S[\varepsilon]/\varepsilon^2$. Prove that the algebra associated to $d|_I$ is isomorphic to $S/I \otimes_{\Bbbk} \frac{\Bbbk[\varepsilon]}{\varepsilon^2}$.

(c) Let $T^1_{S/I}$ be the quotient of $\mathrm{Hom}_S(I, S/I)$ by the space spanned by all $d|_I$. This is an S/I-module called the *first Schlessinger functor*. Compute $T^1_{\Bbbk[x,y]/(xy)}$.

(d) Let $f \in S$ be a element. Prove that $T^1_{S/(f)}$ is zero if and only if the scheme $S/(f)$ is regular. *You can assume $\Bbbk$ algebraically closed, by descent.*

Exercise 2 (Automorphisms of $\mathbb{P}^1_S$)**.** In this exercise S is a $\Bbbk$-scheme and $\mathcal{O}(1)$ on $\mathbb{P}^1 \times S$ is the usual line bundle, defined for example as the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^1_{\Bbbk}$.

(a) Let $\Bbbk$ be a field, and $[a_{00} : a_{01}], [a_{10} : a_{11}], [a_{20} : a_{21}] \in \mathbb{P}^1(\Bbbk)$ be pairwise distinct points. Let $\Delta_{12} = a_{10}a_{21} - a_{11}a_{20}$ and $\Delta_{02} = a_{00}a_{21} - a_{01}a_{20}$. Consider the class

$$\begin{bmatrix} a_{00}\Delta_{12} & -a_{10}\Delta_{02} \\ a_{01}\Delta_{12} & -a_{11}\Delta_{02} \end{bmatrix} \in \mathrm{PGL}_2(\Bbbk) \tag{1.1}$$

Prove that it is well defined and that the associated automorphism σ of $\mathbb{P}^1_{\Bbbk}$ satisfies $\sigma([1:0]) = [a_{00} : a_{01}]$, $\sigma([0:1]) = [a_{10} : a_{11}]$, $\sigma([1:1]) = [a_{20} : a_{21}]$.

(b) Let $\varphi\colon \mathbb{P}^1 \times S \to \mathbb{P}^1 \times S$ be an automorphism. Suppose that $\varphi|_{[1:0]\times S}$ is the identity. Prove that $\varphi^*(\mathcal{O}(1)) \simeq \mathcal{O}(1)$.
Remark: this was sketched during last exercises. Think: line bundles and divisors.

(c) In the setup of (b), suppose further that $\varphi|_{[0:1]\times S}$ and $\varphi|_{[1:1]\times S}$ are identities. Prove that

$$\varphi\colon \mathbb{P}(H^0(\mathcal{O}(1))) \to \mathbb{P}(H^0(\mathcal{O}(1)))$$

is the identity map. Conclude that φ is the identity. *Hint: can assume that S is affine. Then do what you would do for $S = \mathrm{Spec}(\Bbbk)$. If you prefer, you can first assume other nice properties of S, such as being a domain or finite type over $\Bbbk$.*

(d) Let S be a scheme and consider an automorphism $\varphi\colon \mathbb{P}^1 \times S \to \mathbb{P}^1 \times S$. Let

$$\varphi_{[1:0]} = \varphi([1:0] \times (-))\colon S \to \mathbb{P}^1 \times S$$

be its restriction. It is given by $[a_{00} : a_{01}]$, where a_{00}, a_{01} are sections of a line bundle on S, see [15.2.A, 15.2.2, Vakil]. Define $\varphi_{[0:1]}$, a_{10}, a_{11}, and $\varphi_{[1:1]}$, a_{20}, a_{21} analogously. Prove that in this generality the formula (1.1) still makes sense and defines an element $\sigma\colon S \to \mathrm{PGL}_2$.

(e) Continuing with the notation from (d), use (c) to prove that $\varphi(p, s) = (\sigma(s)(p), s)$. Hence, the automorphism φ comes from an element of $\mathrm{PGL}_2(S)$ and $\mathrm{Aut}_S(\mathbb{P}^1_S) \simeq \mathrm{PGL}_2(S)$.