Rozcinanie spodni, odwracanie macierzy: działania $\mathbb{C}^*$ i ich niezmienniki

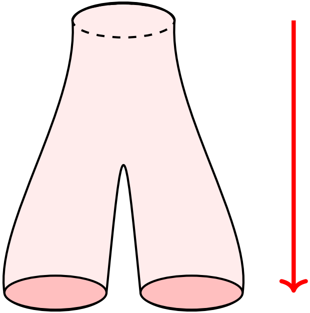
Cutting pants and matrix inversion: $\mathbb{C}^*$ actions and invariants

1882-1969

Wykład im. Wacława Sierpińskiego

Rozcinanie spodni, odwracanie macierzy: działania $\mathbb{C}^*$ i ich niezmienniki

Cutting pants and matrix inversion: $\mathbb{C}^*$ actions and invariants



$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \det A_{11} & \cdots & \pm \det A_{n1} \\ \vdots & & \vdots \\ \pm \det A_{1n} & \cdots & \det A_{nn} \end{bmatrix}$$

Abstract: I will explain how one can resolve (some) rational maps of complex algebraic varieties via Chow quotients of $\mathbb{C}^*$ actions. Based on ideas of Reid et al, Włodarczyk et al, joint work with Michałek, Monin, Romano, Occhetta, Solá Conde.

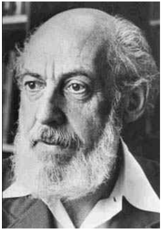
Sierpiński lectures

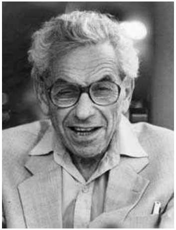

1974


1975


1976


1978


1991


1992

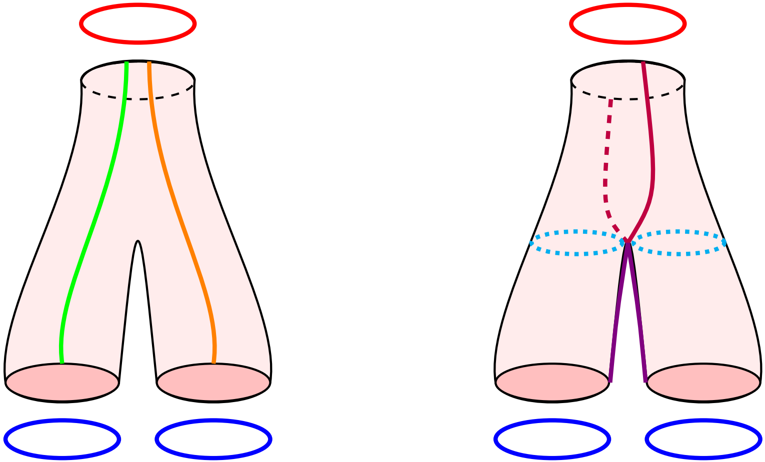

1999


2005

dedicated to ASBB

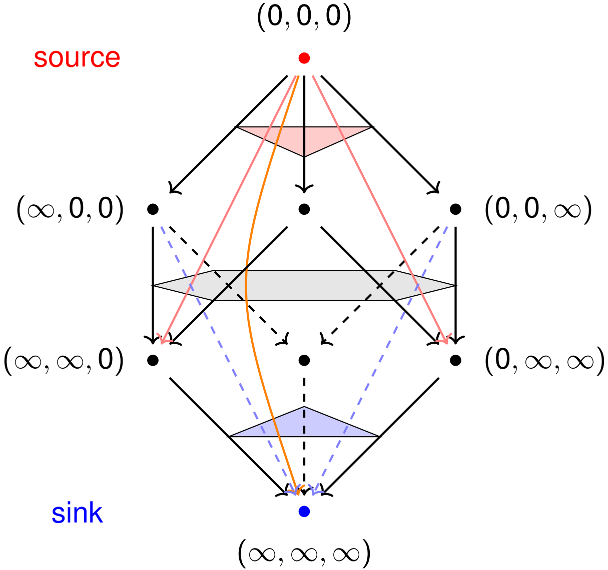
źródło: Wikipedia, MacTutor, University of St. Andrews, Zala Films, arch. prywatne

drops on pants: Morse Theory



Thom: catastrophe theory, Morse: critical points, surgery

drops on a cube



analogy to Morse theory: gradient field → orbits, critical points → fixed points

Cremona transformation, matrix inversion

Take classical Cremona transformation:

$$\mathbb{P}^2 \ni [z_0, z_1, z_2] \longrightarrow [z_1 z_2, z_0 z_2, z_0 z_1] = [z_0^{-1}, z_1^{-1}, z_2^{-1}] \in \mathbb{P}^2$$

Take product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with non-homogeneous coordinates (z_0, z_1, z_2) and $\mathbb{C}^*$ action, with $t \in \mathbb{C}^*$:

$$\boxed{t \cdot (z_0, z_1, z_2) \longrightarrow (tz_0, tz_1, tz_2)}$$

If $z_i \neq 0, \infty$ for $i = 0, 1, 2$ then

$$\lim_{t\to 0} t(z_0, z_1, z_2) = (0, 0, 0) \qquad \lim_{t\to\infty} t(z_0, z_1, z_2) = (\infty, \infty, \infty)$$

$$\frac{\partial t(z_0,z_1,z_2)}{\partial t}\Big|_{t=0} = (z_0, z_1, z_2) \qquad \frac{\partial t(z_0,z_1,z_2)}{\partial t}\Big|_{t=\infty} = (z_0^{-1}, z_1^{-1}, z_2^{-1})$$

So we have a description of Cremona in terms of $\mathbb{C}^*$ action:

tangent to general orbit at 0 $\longrightarrow$ tangent to general orbit at ∞

how we tell students to invert matrices

You can invert invertible matrices only!

1st method:
a polynomial formula

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\frac{1}{\det A} \begin{bmatrix} \det A_{11} & \cdots & \pm \det A_{n1} \\ \vdots & & \vdots \\ \pm \det A_{1n} & \cdots & \det A_{nn} \end{bmatrix}$$

2nd method:
reduction to echelon form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

where $A^{-1} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

$\mathbb{C}^*$ action on Grassmanians

Rows $n \times 2n$ matrix represent n vectors in a space V of dim $2n$. Multiplication from left by $n \times n$ invertible matrix does not change the linear space spanned by them. Therefore

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

is the equality on a Grassmann variety $Grass(n, V)$ parametrizing n-subspaces of V.

We write $V = W_1 \oplus W_2$ as a sum of two n-spaces and define $\mathbb{C}^*$ action:

$$\mathbb{C}^* \times V \ni (t, w_1 + w_2) \longrightarrow tw_1 + w_2 \in V$$

The action lifts to $Grass(n, V)$ and for a general n-space $W \subset V$

$$\lim_{t \to 0} t \cdot [W] = [W_2] \text{ and } \lim_{t \to \infty} t \cdot [W] = [W_1]$$

while the tangents to the orbit change as above $A \leftrightarrow A^{-1}$.

the punchline

(1) Inversion of matrices is related to $\mathbb{C}^*$ action and spaces of orbits.

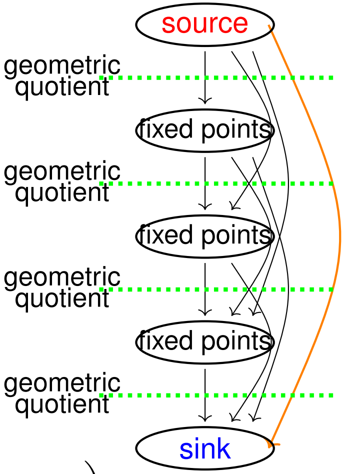
(2) ABB: analogy to Morse theory: sections = geometric quotients = parameter spaces for (almost all) orbits

(3) Mumford: take ample line bundle L, linearization

$$\mathrm{H}^0(X, L) = \bigoplus_{\mu} \mathrm{H}^0(X, L)^{\mu}$$

defines

$$X /\!\!/^{\mu} \mathbb{C}^* = \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)^{m\mu} \right) = \mathcal{Y}_{\mu}$$



GIT paradigm: quotients defined on open subsets

birational maps via cobordism

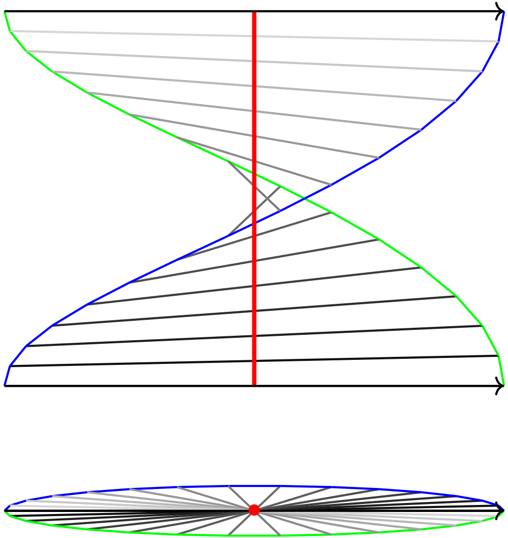
Definition A birational map of algebraic varieties $X_1 \leftarrow\dashrightarrow X_2$ is a bijective function defined on Zariski open subsets of X_i's which locally is described by rational functions.

Example Inversion of matrices defines a birational map

$$\mathbb{P}(M_n(\mathbb{C})) \dashrightarrow \mathbb{P}(M_n(\mathbb{C}))$$

Theorem [Włodarczyk, 2000]
For any birational map $X_1 \leftarrow\dashrightarrow X_2$ of smooth algebraic varieties there exists algebraic cobordism i.e. a smooth variety with $\mathbb{C}^*$ action whose two geometric quotients are X_1 and X_2.

the blow-up: an elementary birational modification



Hironaka's conjecture, Włodarczyk's theorem

Hironaka Strong Factorisation Conjecture
Any birational map of smooth varieties can be resolved via a sequence of consecutive blow-ups followed by blow-downs in smooth centers:

$$
\begin{array}{ccccccccccccc}
 & & & & & & Y_{n,m} & & & & & & \\
 & & & & \dots & \swarrow & & \searrow & \dots & & & & \\
 & & Y_{n,1} & \swarrow & & & & & & \searrow & Y_{1,m} & & \\
X_{n,0} & \swarrow & & & & & & & & & & \searrow & X_{0,m} \\
X_{n,0} & & & & & & \dashleftarrow\!\dashrightarrow & & & & & & X_{0,m}
\end{array}
$$

Theorem [Abramovich, Karu, Matsuki, Włodarczyk] Any birational map of smooth varieties can be factored into a sequence of birational maps each of them resolved by blow-up and blow-down:

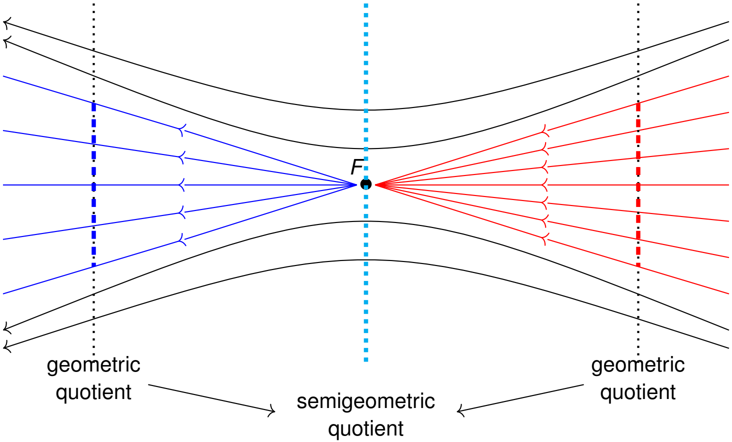
$$
\begin{array}{ccccccccc}
 & Y_1 & & Y_2 & & & & Y_{n-1} & \\
\swarrow & & \searrow\ \swarrow & & \searrow & & \swarrow & & \searrow \\
X_1 & \dashleftarrow\!\dashrightarrow & X_2 & \dashleftarrow\!\dashrightarrow & \dots & & \dashleftarrow\!\dashrightarrow & & X_n
\end{array}
$$

Idea Use algebraic cobordism, X_i's are geometric quotients.

algebraic local surgery: flips and flops

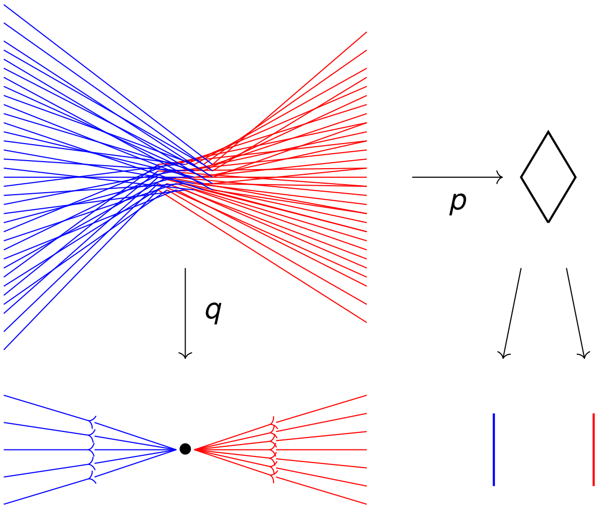
Thaddeus, Reid, Dolgachev, Hu, 1990's

Variation of geometric quotients VGIT



resolving flips/flops

Modification can be resolved locally by taking unions of closures of orbits



Chow/Hilbert/universal quotients

Fujiki [1979], Białynicki-Birula, Sommese [1983], Kapranov [1992]

Given the action $\mathbb{C}^* \times X \to X$ consider the maximal family of invariant 1-cycles containing a general orbit as a general point, by $\mathcal{C}$ we denote its normalization and $\mathcal{U}$ the universal family

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{p} & \mathcal{C} \\ \downarrow{\scriptstyle q} & & \\ X & & \end{array}$$

a new paradigm for the quotient

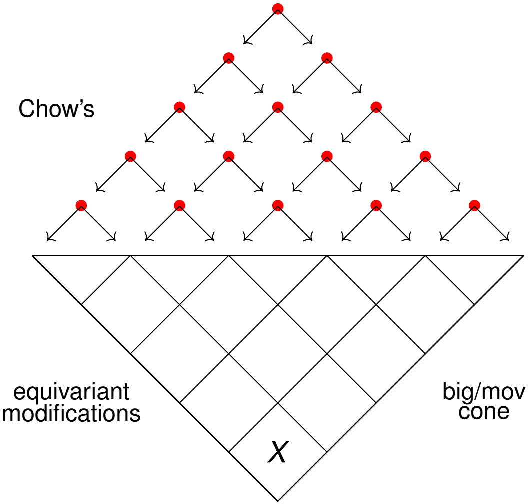
If the morphism p is flat then we have equivariant decomposition of a vector bundle over $\mathcal{C}$

$$p_* q^*(mL) = \bigoplus_{u \in \mathbb{Z}} \mathcal{L}_m^u$$

with $\mathrm{H}^0(\mathcal{C}, \mathcal{L}_m^u) = \mathrm{H}^0(X, mL)^u$ hence we have regular morphisms to GIT quotients

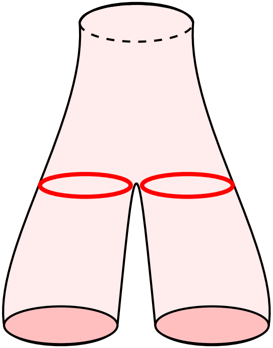
$$\mathcal{C} \longrightarrow \mathcal{Y}_u$$

big picture: equivariant modifications, Chow quotients

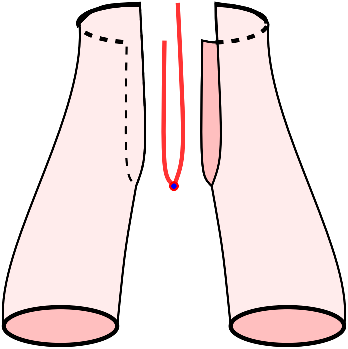


lower part: chambers in big cone; upper part: ● represent normalized Chow quotients for resp. equiv. modification; bottom row ● are geom. quotients

cutting trousers



In topology you cut pants on level function.

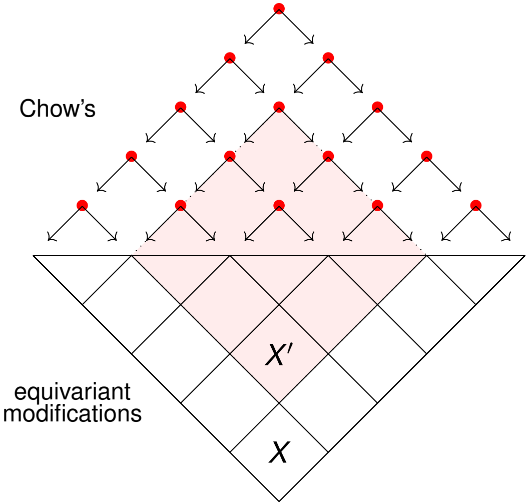


In algebraic geometry cut pants along BB cells.

Equivariant modification, surgery along orbits, equivalently:

$$\mathrm{H}^0(X, L) = \bigoplus_{\mu} \mathrm{H}^0(X, L)^{\mu} \rightsquigarrow \mathrm{H}^0(X', L') = \bigoplus_{\mu \in [a,b]} \mathrm{H}^0(X, L)^{\mu}$$

cutting trousers, changing quotients



GIT/Chow quotients after equivariant modifications;
pink region: quotients for X'

why constructing resolution?

Question Given a birational map $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ find the degree of $\overline{\varphi(\Lambda)}$ with $\Lambda \subset \mathbb{P}^N$ a general linear subspace of dimension r.

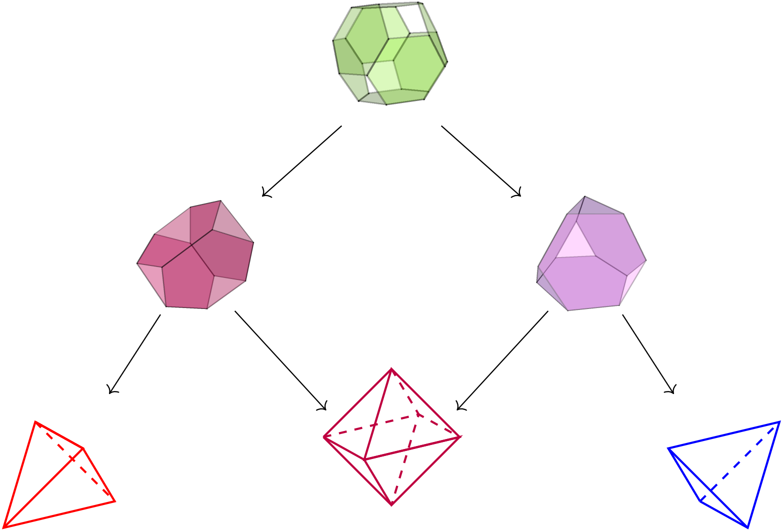
If φ is the inversion map for $n \times n$ diagonal matrices then we get Newton binomial coefficient.

If φ is the inversion map for $n \times n$ symmetric matrices then coefficients for $n = 3, \ldots, 6$ are

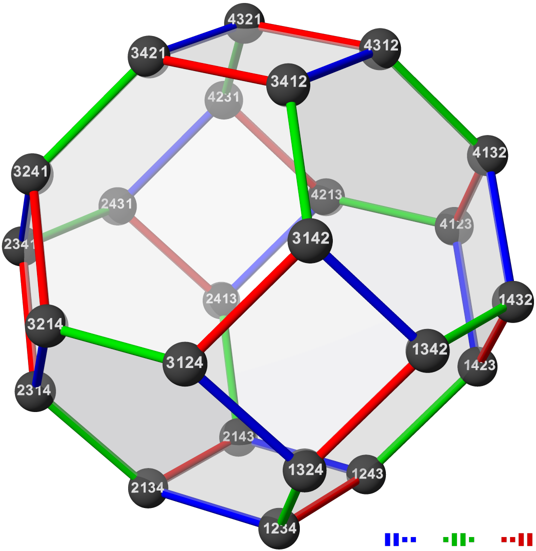
1	2	4	4	2	1					
1	3	9	17	21	21	17	9	3	1	
1	4	16	44	86	137	188	212	188	137	86
1	5	25	90	240	528	1016	1696	2396	2886	3054

And this is apparently useful in statistics for calculating ML degree.

inverting 4×4 diagonal matrices on picture



Chow quotient: permutahedron



source: Wikipedia