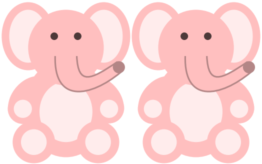
Two elephants (almost) imply LeBrun–Salamon conjecture



based on work of Buczyński, Weber

Romano, Occhetta, Solá Conde, and Śmiech

the notion of an elephant as an element in anticanonical system is due to Miles Reid

using tikz and tikzlings for pictures

LeBrun–Salamon conjecture

- The conjecture in Riemannian differential geometry: positive quaternion-Kähler manifolds are symmetric spaces (Wolf spaces).
- The conjecture in complex algebraic geometry: every Fano complex contact manifold is homogeneous and in fact the projectivisation of the minimal (co)adjoint orbit of a simple group.

Algebraic geometry conjecture is stronger. A slightly weaker version with assumption of existence of Kähler-Einstein metric implies diff geom version too. In particular one may assume that the group of automorphisms is reductive.

(co)adjoint orbits

Orbits of the co-adjoint representation of *G* admit Kostant- -Kirillov *G* invariant symplectic form ω on an orbit $\mathcal{O} \subset \mathfrak{g}^*$

$$\omega_\nu(ad^*_\xi(\nu), ad^*_\eta(\nu)) = \nu([\xi,\eta]) \quad \text{for} \quad \xi,\eta \in \mathfrak{g},\ \nu \in \mathcal{O}$$

The symplectic form ω descends to a contact form θ on $\mathbb{P}(\mathcal{O})$; for *G* simple co-adjoint can be identified with adjoint.

Dimension of the projective minimal adjoint orbits for simple groups

A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$2r-1$	$4r-5$	$2r-1$	$4r-7$	21	33	57	15	5

Fano contact manifolds

- Let L be an ample line bundle on a complex manifold X, $\dim X = 2n+1$, a contact form $\theta \in \mathrm{H}^0(X, \Omega_X \otimes L)$ is such that $(d\theta)^{\wedge n} \wedge \theta$ nowhere vanishes; this implies

$$-K_X = (n+1)L$$

- Let F be the kernel of $\theta : TX \to L$ then $d\theta$ defines nondegenerate skew-symmetric pairing:

$$d\theta : F \times F \to L$$

- Partial results on contact and quaternion-Kähler manifolds: small $\dim X$, big torus, L has many sections
- We may assume $\mathrm{Pic}\, X = \mathbb{Z} \cdot L$, otherwise

$$(X, L) = (\mathbb{P}^{2n+1}, \mathcal{O}(2)) \text{ or } (\mathbb{P}(T\mathbb{P}^{n+1}), \mathcal{O}(1))$$

[Hitchin, Poon, LeBrun, Salamon, Herrera^2, Bielawski, Fang, Druel, Beauville, Kebekus, Peternell, Sommese, W, Demailly]

a diagram

We use Atiyah extension defined by $c_1(L) \in Ext(\mathcal{O}, \Omega X)$

$$0 \longrightarrow \Omega X \longrightarrow \mathcal{L} \longrightarrow \mathcal{O} \longrightarrow 0$$

to get the following commutative diagram

$$\begin{array}{ccccc}
F & & & & \\
\downarrow & & & & \\
TX & \longleftarrow & \mathcal{L}^* & \longleftarrow & \mathcal{O} \\
\theta \downarrow & & \cong \downarrow \widehat{\omega} & & \downarrow \theta \\
L & \longleftarrow & \mathcal{L} \otimes L & \longleftarrow & \Omega X \otimes L \\
 & \underset{d}{\dashrightarrow} & & & \downarrow \\
 & & & & F^* \otimes L
\end{array}$$

where $\mathcal{L} \otimes L$ are first jets and d is differentiation of sections and the dotted arrow is $d\theta$.

sections of L and $\mathfrak{g}$

Thus we have a map $\mathrm{H}^0(X, L) \to \mathrm{H}^0(X, TX)$ splitting θ with image being the Lie algebra $\mathfrak{g}$ of group G of automorphisms preserving contact structure; in fact $G = Aut(X)$ unless $X = \mathbb{P}^{2n+1}$ [Beauville, Kebekus].

We may assume that G is reductive hence (up to finite cover) a product of simple groups and algebraic torus and the ajoint representation of G on $\mathfrak{g} = \mathrm{H}^0(X, L)$. Let H be the maximal torus in G; $\mathrm{rk} G = \mathrm{rk} H$.

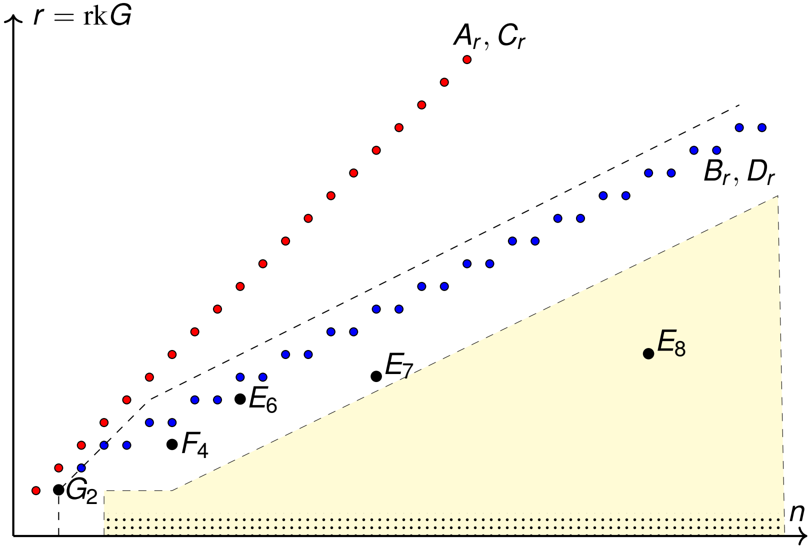
Theorem

Suppose that X is a Fano contact manifold of dimension $2n + 1$ with $G = Aut(X)$ reductive. If $n \leq 4$ or

$$\mathrm{rk}(G) \geq \max(2, (n-3)/2)$$

then G is simple and X is the minimal adjoint orbit in $\mathbb{P}(\mathfrak{g})$.

[BWW], [ORSW]: contact manifolds, torus action



yellow region is where neither [BWW],[ORSW] works, dotted where $\mathrm{rk}(G) = 1$

the main technical result and

Theorem (ORSW)

Suppose that X is a Fano contact manifold of dimension $2n+1$ with $G = Aut(X)$ reductive. If $\mathrm{rk} G \geq 2$ and all extremal fixed point components of the action of H are isolated points then G is simple and X is the minimal adjoint orbit in $\mathbb{P}(\mathfrak{g})$.

Two elephant conjecture For every Fano manifold Y of dimension ≥ 1 with an ample line bundle L such that $\mathit{Pic} Y = \mathbb{Z} \cdot L$, it holds

$$\dim \mathrm{H}^0(Y, L) \geq 2$$

Theorem (Ś)

Suppose the above conjecture holds for every Fano manifold and X is a Fano contact manifold with $G = Aut(X)$ reductive. Then either X is the minimal adjoint orbit in $\mathbb{P}(\mathfrak{g})$ or $G = SL(2)$.

idea: polytope of section

Let H be an algebraic torus with $M = \mathrm{Hom}(H, \mathbb{C}^*) \simeq \mathbb{Z}^r$.
Assume that H acts (effectively) on a polarized manifold (X, L) (with linearization $\mu : H \times L \to L$).

- We have decomposition of space of sections into eigenspaces
$$\mathrm{H}^0(X, L) = \bigoplus_{u \in M} \mathrm{H}^0(X, L)^u$$
- By $\Gamma(L)$ we denote convex hull in $M_{\mathbb{R}}$ of eigenvalues (weights) of the action of H on $\mathrm{H}^0(X, L)$.
- For (X, L) contact with $\mathrm{H}^0(X, L) = \mathfrak{g}$ this decomposition is the same as for the adjoint action of $H \subset G = Aut(X)$ on $\mathfrak{g}$ and the polytope $\Gamma(L)$ is the same as root polytope $\Delta(G)$.

idea: polytope of fixed points

- We have decomposition of the set of fixed points

$$X^H = Y_1 \sqcup \cdots \sqcup Y_s$$

- By $\widetilde{\Delta}(X, H, L, \mu) \subset M$ we denote the set of weights $\mu(Y_i)$ of the action of H on fibers of L over Y_i's and by $\Delta(L)$ their convex hull in $M_{\mathbb{R}}$.
- A connected component $Y \subset X^H$ is called *extremal* if $\mu(Y)$ is a vertex of Δ.

Comparing polytopes:

- $\Delta(L^{\otimes m}) = m \cdot \Delta(L)$ and $\Gamma(L^{\otimes m}) \supseteq m \cdot \Gamma(L)$
- $\Delta(L) = \Gamma(L)$ if L is base point free hence $\Gamma(L) \subseteq \Delta(L)$

local action information, the compass

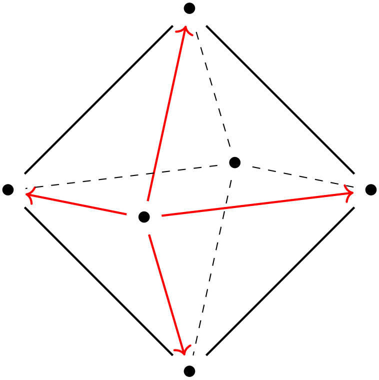
Let $Y \subset X^H$ be a connected component.
Take $y \in Y$ and consider the action of H on $T_y^* X$: it splits into eigenspaces associated to some characters (weights) in M; the trivial eigenspace is $T_y^* Y$

The set non-zero (multiple) weights of this action is called *the compass* of the action of H on the component Y and we denote it $\mathcal{C}(Y, X, H)$

example: 4-dimensional quadric

Three-dimensional torus acting on the 4-dimensional quadric $x_1x_2 + x_3x_4 + x_5x_6 = 0$ with weights

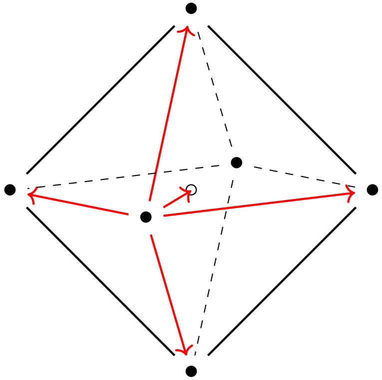
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$



Six fixed points, six sections, four elements in the compass

example 5-dimensional quadric

Three-dimensional torus acting on the 5-dimensional quadric
$x_0^2 + x_1 x_2 + x_3 x_4 + x_5 x_6 = 0$



Six fixed points, seven sections, five elements in the compass.

a grid is a generalization of torus moment polytope and a GKM graph

downgrading and reduction, 1

Consider a sequence of tori

$$0 \longrightarrow H_1 \xrightarrow{\pi} H \xrightarrow{\iota} H_2 \longrightarrow 0$$

and the associated sequence of lattices of characters

$$0 \longrightarrow M_2 \xrightarrow{\iota} M \xrightarrow{\pi} M_1 \longrightarrow 0$$

We have the action of H_2 on components of X^{H_1} and for every connected component $Y_1 \subset X^{H_1}$ we get

$$Y_1^{H_2} = X^H \cap Y_1$$

Note: for a general choice $H_1 \hookrightarrow H$ we have $X^{H_1} = X^H$

downgrading and reduction, 2

The restriction of the action to $H_1 \hookrightarrow H$ implies

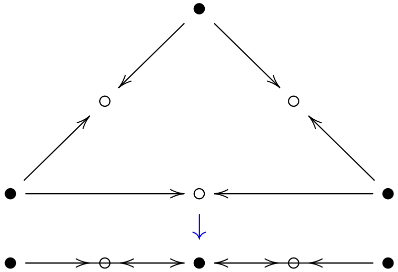
$$\pi(\Delta(X, L, H, \mu)) = \Delta(X, L, H_1, \mu_{H_1})$$

For every pair of connected components $Y_1 \subset X^{H_1}$ and $Y \subset Y_1^{H_2}$ we have

- the elements of $\mathcal{C}(Y_1, X, H_1)$ are π-projections of elements from $\mathcal{C}(Y, X, H)$
- the elements of $\mathcal{C}(Y, Y_1, H_2)$ are those in $\mathcal{C}(Y, X, H)$ which are in the kernel of π

example: downgrading $\mathbb{P}^2$

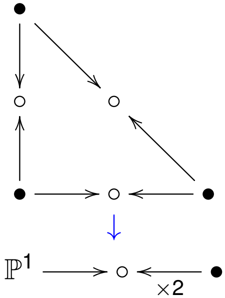
Downgrading $(\mathbb{C}^*)^2$ acting on $\mathbb{P}^2$ with $\mathcal{O}(2)$:



to $\mathbb{C}^*$ acting with weights $(0, 1, 2)$.

example: downgrading $\mathbb{P}^2$

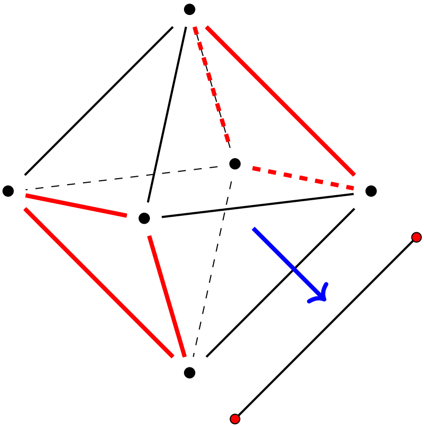
Downgrading $(\mathbb{C}^*)^2$ acting on $\mathbb{P}^2$ with $\mathcal{O}(2)$:



to $\mathbb{C}^*$ acting with weights $(0, 0, 1)$.
Note that quotient torus acts on $\mathbb{P}^1$ of fixed points.

4-dimensional quadric: downgrading the action

Downgrading to one dimensional torus acting on the 4-dimensional quadric with two fixed point components $\simeq \mathbb{P}^2$.



Note the quotient torus acting.

BB decomposition

For $H = \mathbb{C}^*$ with coordinate t and X projective manifold we have Białynicki-Birula decomposition:

- Take decomposition $X^H = Y_1 \sqcup \cdots \sqcup Y_s$ and for every Y_i by $\nu^{\pm}(Y_i)$ denote the positive and negative number of characters in the compass.
- Define
  $X_i^+ = \{x \in X : \lim_{t\to 0} t \cdot x \in Y_i\}$
  $X_i^- = \{x \in X : \lim_{t\to \infty} t \cdot x \in Y_i\}$
- Then
  - $X = X_1^+ \sqcup \cdots \sqcup X_s^+ = X_1^- \sqcup \cdots \sqcup X_s^-$,
  - partial order $Y_i \prec Y_j \Leftrightarrow \overline{X_i^+} \supset Y_j$ agrees with $\mu(Y_i) < \mu(Y_j)$
  - the unique dense $^{\pm}$-component is called source/sink,
  - $X_i^{\pm} \to Y_i$ is a $\mathbb{C}^{\nu^{\pm}(Y_i)}$ fibration,
  - $H_m(X, \mathbb{Z}) = \bigoplus_i H_{m-2\nu^+(Y_i)}(Y_i, \mathbb{Z}) = \bigoplus_i H_{m-2\nu^-(Y_i)}(Y_i, \mathbb{Z})$

BB decomposition, case Pic $\simeq \mathbb{Z}$

Assume in addition that $\operatorname{Pic} X = \mathbb{Z} \cdot L$ and $Y_0 \subset X^H$ is the source. Then X Fano and one of the following holds:

1. $\dim Y_0 > 0$ and
   - Y_0 is Fano with $\operatorname{Pic} Y_0 = \mathbb{Z} \cdot L$,
   - the complement of X_0^+ is of codimension ≥ 2,
   - $\mathrm{H}^0(X, L) \to \mathrm{H}^0(Y_0, L)$ is surjective.
2. Y_0 is a point and
   - X_0^+ is an affine space
   - $D = X \setminus X_0^+$ is an irreducible divisor in the system $|L|$,
   - there exists the unique fixed point component $Y_1 \subset X^H$ such that $\mu(Y_1)$ is minimal in $\widetilde{\Delta}(X, L, H, \mu) \setminus \mu(Y_0)$,
   - X_1^+ associated to Y_1 is dense in D.

Corollary:  $\Longrightarrow$ isolated extremal points

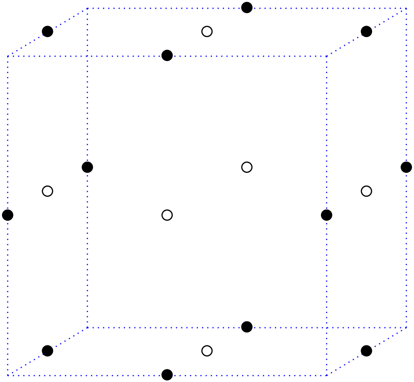
First note  + BB $\Longrightarrow \Gamma(L) = \Delta(L) = \Delta(Aut(X))$.

reduction to $SL(2)$ action

- $G = Aut(X)$ is reductive $\Longrightarrow$
  - $\Rightarrow$ G is a product of a semisimple and a torus
  - $\Rightarrow$ G has no torus factor: the action on $\mathfrak{g} = \mathrm{H}^0(X, L)$ is adjoint
  - $\Rightarrow$ G is simple: analyze the compass.
- G is simple and Cartan torus action $H < G$ acts with isolated extremal fixed points $\Longrightarrow$ there exists $SL(3) \subset G$ with $H_2 < SL(3)$ acting with isolated extremal points.

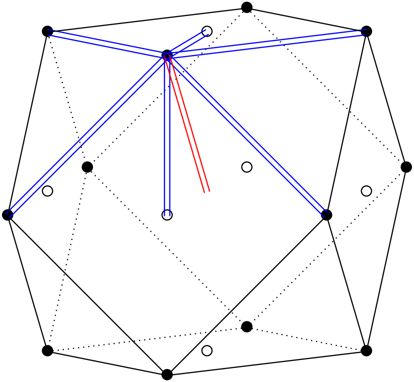
example: minimal nilpotent orbit of B_3

B_3 root system



example: minimal nilpotent orbit of B_3

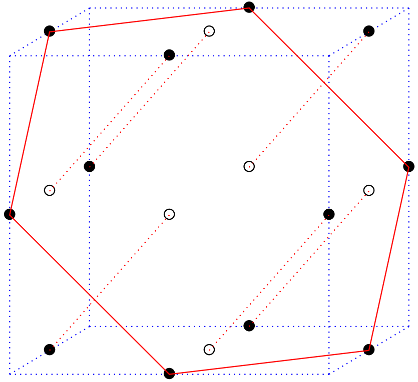
Root polytope of B_3 and the compass.



note the symmetry in the compass induced by $d\theta$

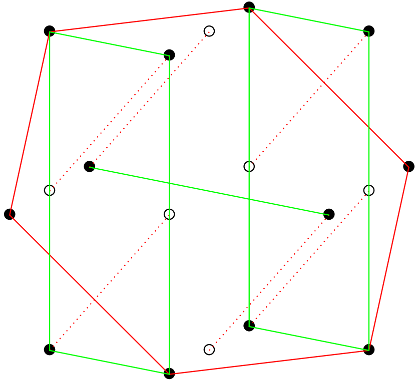
example: minimal nilpotent orbit of B_3

Downgrading the action.



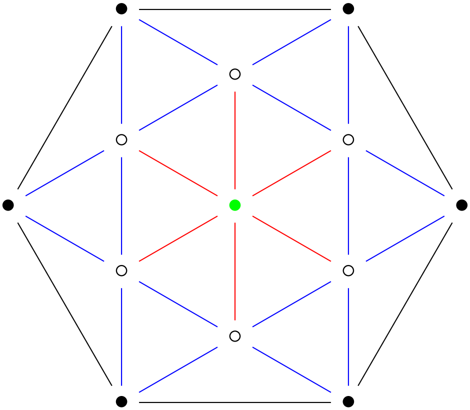
example: minimal nilpotent orbit of B_3

Downgrading and restricting the action



finding fixed point components

Position of the fixed points components in terms of μ:



Starting point: understand varieties with $\mathbb{C}^*$ action associated to blue line segments and next to red segments.

bandwidth 3 varieties

Let (X, L) be a variety with $\mathbb{C}^*$ action. The bandwidth of the action is $\mu(Y_0) - \mu(Y_\infty)$, where Y_0, Y_∞ are source and sink of the action and μ is any linearization of L.

Theorem (*)

Let (X, L) be a variety of $\dim \geq 3$ with $\mathbb{C}^*$ action of bandwidth 3 with sink and source isolated points and no non-trivial isotropy group. The (X, L) is one of the following:

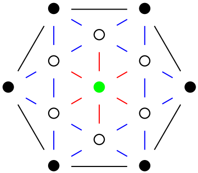
- *(X, L) is a scroll over $\mathbb{P}^1$*
- $(X, L) = (\mathbb{Q}^{n-1} \times \mathbb{P}^1, \mathcal{O}(1, 1))$
- *X is rational homogeneous of type $C_3(3)$, $A_5(3)$, $D_6(6)$, $E_7(7)$, and L is the generator of $\operatorname{Pic} X$.*

The varieties in the last case are VMRT's for adjoint orbits for simple groups F_4, E_6, E_7 and E_8.

* ORSW based on Cremona transformation results by Zak and Ein, Shepherd-Barron

comparig the fixed point data

Understanding varieties with bandwidth 3 (blue segments) and of bandwidth 2 (red segments) we recover fixed point components and their compases



Corollary

For every contact variety (X, L) such that the action of $H \subset Aut(X) = G$ has isolated extremal fixed points the action of a chosen two dimensional torus $H_2 \subset H$ has the same fixed point components (as polarized varieties) and compasses (as normal bundles with H_2 action) as in case of the action of H_2 on the minimal adjoint orbit $X_G \subset \mathbb{P}(\mathfrak{g})$.

a tool: equivariant cohomology

Grothendieck-Atiyah-Bott-Berline-Vergne localization in cohomology and Riemann-Roch theorem (simplest version):

Assume that X^H consists of isolated points $y_1, y_2, \dots y_k$. Take $\mu_i = \mu(y_i)$ and $\nu_{i,j}$ are elements of $\mathcal{C}(y_i, X, H)$.
Then the character of the representation of H on $\mathrm{H}^0(X, L^{\otimes m})$ is equal

$$\sum_{i=1}^{k} \frac{t^{m\mu_i}}{\prod_j (1 - t^{\nu_{i,j}})}$$

Corollary

Suppose that a simple group G with a maximal torus H acts on X, $\operatorname{Pic} X = \mathbb{Z}L$*, so that the data* $\mu_i,\ \nu_{i,j}$ *is the same as for a G-homogeneous manifold* $\widehat{X}$*,* $\operatorname{Pic} \widehat{X} = \mathbb{Z}\widehat{L}$*. Then*

$$(X, L) = (\widehat{X}, \widehat{L})$$

conclusion: reverse engineering

As above, (X, L) contact with torus $H \subset Aut(X) = G$, where G is a simple group with $(X_G, \mathcal{O}(1)) \subset (\mathbb{P}(\mathfrak{g}), \mathcal{O}(1))$ adjoint variety. By torus action data (a grid) we understand the set of all fixed components $(Y_i, L_{|Y_i})$ and isomorphism classes of their normal bundles with torus action.

The scheme of the proof, conclusion:

- The torus action data for (X, L) and $(X_G, \mathcal{O}(1))$ is the same.
- $\mathrm{H}^0(X, mL) \cong \mathrm{H}^0(X_G, \mathcal{O}(m))$ as representations of H.
- $\mathrm{H}^0(X, mL) \cong \mathrm{H}^0(X_G, \mathcal{O}(m))$ as representations of G.
- $(X, L) \cong (X_G, \mathcal{O}(1))$ as G-varieties.