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A report on Fano manifolds of middle index and $b_2 \geq 2$.

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Let X be a smooth projective variety of dimension n defined over complex numbers. We say that X is Fano if its anti-canonical divisor $-K_X$ (or, equivalently, its first Chern class c_1X) is ample. The index $r = r(X)$ of the Fano manifold X is defined as the largest integer dividing $-K_X$ in the Picard group $PicX \cong \mathbf{Z}^\rho$. The task of this report is to present a classification of Fano manifolds of index r and dimension $n = 2r$ whose second Betti number $b_2(X) = \rho$ is at least 2.

(1.1). It is known that the index r of a Fano manifold X is *at most* $dimX + 1$. Moreover if either $r = dimX + 1$ or $r = dimX$ then X is a either the projective $\mathbf{P}^n$ or the smooth quadric $\mathbf{Q}^n$, respectively (see [KO]). If X is a Fano surface of index 1 then it is called del Pezzo surface and is obtained by blowing-up $b_2(X) - 1$ general points on $\mathbf{P}^2$ (where $2 \leq b_2(X) \leq 8$). The degree $d(X) = (-K_X)^2$ of a del Pezzo surface X is then equal to $9 - b_2(X)$. The complete linear system $|-K_X|$ is base point free for $d = d(X) \geq 2$ and gives a map of X into $\mathbf{P}^d$; for $d \geq 3$ this is an embedding.

(1.2). Fano manifolds of of dimension $n \geq 3$ and index $n-1$ are also called del Pezzo manifolds and were classified by Fujita, see for example [F1, I.8.11]. From Fujita's classification it follows that, if X is a del Pezzo manifold of dimension $n \geq 4$ then it is one of the following:

- $V_1 =$ a weighted hypersurface of degree 6 in the weighted projective space $\mathbf{P}(1, \ldots, 1, 2, 3)$,
- $V_2 =$ a weighted hypersurface of degree 4 in the weighted projective space $\mathbf{P}(1, \ldots, 1, 2, 2)$ (equivalently: a double covering of $\mathbf{P}^n$ ramified along quartic),
- $V_3 =$ a hypercubic in $\mathbf{P}^{n+1}$,
- $V_4 =$ a complete intersection of type (2,2) in $\mathbf{P}^{n+2}$,
- $V_5 =$ a linear section of Grasmanian $G(2, W) \subset \mathbf{P}(\Lambda^2 W)$ of linear spaces of dimension 2 in a 5-dimensional linear space W, $dimV_5 \leq 6$,
- $V_6 = \mathbf{P}^2 \times \mathbf{P}^2$.

In dimension 3 we have also $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{P}(T\mathbf{P}^2)$ where $T\mathbf{P}^2$ denotes the tangent bundle to $\mathbf{P}^2$, and $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2) \oplus \mathcal{O}_{\mathbf{P}^2}(1))$.

The subscript d of the del Pezzo manifold V_d denotes its degree, that is the self-intersection of the first Chern class of the line bundle $\mathcal{O}_V(1)$ such that $\mathcal{O}_V(1)^{\otimes(n-1)} = \mathcal{O}_V(-K_X)$.

Fano n-folds of index $n-2$ were classified by Mukai [Mu] under an assumption on existence of smooth descending sequence of submanifolds, so-called *smooth ladder*.

(2.1). Based on the above examples and on the classification of Fano n-folds of index $n-2$ Mukai conjectured that if $r(X) > n/2+1$ then $b_2(X) = 1$ and if $r(X) = n/2+1$ and $b_2(X) \geq 2$ then

$$X \cong \mathbf{P}^{r-1} \times \mathbf{P}^{r-1}.$$

The conjecture was proved in [W1] and, subsequently, in [W2] Fano n-folds with $r(X) = (n+1)/2$ and $b_2(X) \geq 2$ were studied. The result was the following

Theorem 2.2. *Let X be a Fano n-fold (where n is odd ≥ 3) of index $r = (n+1)/2$. If $b_2(X) \geq 2$ then X has a $\mathbf{P}^{r-1}$-bundle structure over either projective space $\mathbf{P}^r$ or a smooth quadric $\mathbf{Q}^r$. The structure of X is described as follows:*

No.	*Base*	*projective bundle*	*another description*
1.	$\mathbf{P}^r$	$\mathbf{P}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)})$	*blow-up of* $\mathbf{P}^{r-1} \subset \mathbf{P}^{2r-1}$
2.	$\mathbf{P}^r$	$\mathbf{P}(T\mathbf{P}^r)$	$(1,1)$ *divisor in* $\mathbf{P}^r \times \mathbf{P}^r$
3.	$\mathbf{Q}^r$	$\mathbf{P}(\mathcal{O}(1)^{\oplus r})$	$\mathbf{P}^r \times \mathbf{Q}^r$

The proof od the above result is a two-step argument. First, using the deformations of rational curves contracted by elementary contractions of X one was able to establish that there exists an elementary contraction of X (in terms of Mori theory) whose fibers are of dimension $\leq r-1$. Then, the following result was used.

Proposition 2.3. [F2, Lemma 2.12], [I] *Let $p : X \to Y$ be a map of a smooth projective variety X onto a normal variety Y such that $p_*\mathcal{O}_X = \mathcal{O}_Y$ (a contraction morphism with connected fibers). Assume that on X there exists an ample line bundle L such that $K_X + rL$ and all fibers of p are of dimension $\leq r$. Then Y is smooth and $p : X \to Y$ is a projective bundle over Y, that is, there exists a vector bundle $\mathcal{E}$ over Y such that $X = \mathbf{P}(\mathcal{E})$ (one may assume $\mathcal{E} \cong p_*L$).*

The proposition was used for $L = \mathcal{O}_X(-K_X/r)$ so that, by adjunction, the resulting bundle $\mathcal{E} := p_*L$ was ample of rank $r = dimY$ and $c_1\mathcal{E} = c_1Y$. Studying pairs $(Y, \mathcal{E})$ satisfying these conditions was the second step of the proof of (2.2), see [P1], [P2] and [YZ].

(3.1). Fano manifolds of index r and dimension $n = 2r$ we will call for short Fano n-folds of middle index. By H we will denote the ample divisor such that $rH = -K_X$; the self-intersection of H we call the degree of such X and will be denoted by $d(X)$.

The above mentioned results may suggest that there is possible to classify these Fano manifolds of middle index which have $b_2 \geq 2$. Indeed, similar arguments can be applied. To make the exposition more transparent let us assume that $r \geq 3$ so that X is of dimension ≥ 6, though it should be noted that the arguments can be extended to this case as well, and the result coincides with the classification of Mukai [Mu], see also [W4]. Also, as it was proved in [W5] and in [K], for $n \geq 6$ we may assume that $b_2(X) = 2$ with the only exception in dimension 6 when $X \cong \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$.

Let us recall that an elementary contraction $\varphi : X \to Y$ is a morphism onto a normal variety Y such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y$ and all curves contracted by φ are proportional in $H_2(X, \mathbf{Z})$. Elementary contractions of smooth 3-folds are known due to Mori [M2]. The classification of Fano 3-folds with $b_2 \geq 2$ done by Mori and Mukai [MM] and is based on studying of the elementary contractions. We follow this idea.

A Fano n-fold with $b_2 = 2$ has two elementary contractions. Studying the interplay between the deformations of rational curves contracted by these maps (similarly as in [M1]) allows to derive the following description

Proposition 3.2. [W5, Thm I] *Let X be a Fano manifold of middle index with $b_2 \geq 2$. There exists an elementary contraction $p : X \to Y$ such that*
(i) $dimY < dimX$ and all fibers of p are of dimension $\leq r$.
"The other" contraction $\varphi : X \to Z$ is either of the type (i), or φ is birational and either
(ii) its exceptional set is an irreducible divisor E and $\varphi_{|E}$ has all fibers of dimension r, or
(iii) there exists a fiber of φ of dimension $= r+1$, but then all fibers of p are of dimension $r-1$.

The next ingredient of the proof comes from the study of contraction morphisms

Proposition 3.3. [AW Thm 4.1], [ABW] *Let $p : X \to Y$ be an elementary contraction of a smooth variety X onto a normal variety Y of smaller dimension. Assume that there exists an ample line bundle L on X such that $K_X + rL$ is trivial on fibers of p and all fibers of p are of dimension $\leq r$. Then Y is smooth and one of the following occurs:*
(i) $dimY = r+1$ and $\mathcal{E} := p_*L$ is a locally free sheaf of rank r and $p : X \to Y$ is a projective bundle, $X \cong \mathbf{P}(\mathcal{E})$, (this is the case of (2.3));
(ii) $dimY = r$ and $\mathcal{E} := p_*L$ is a locally free sheaf of rank $r+2$ and $p : X \to Y$ is a quadric bundle so that X can be embedded as a divisor of relative degree 2 into $\mathbf{P}(\mathcal{E})$;
(iii) $dimY = r+1$ and $\mathcal{E} := p_*L$ is a reflexive non-locally free sheaf of rank r with isolated singularities and $X \cong \mathbf{P}(\mathcal{E})$.

(3.4). The singularities of the non-locally free sheaf $\mathcal{E}$ occuring in the last case of the above proposition are very special. Namely, locally, at a point $y \in Y$ we have an exact sequence:

$$0 \longrightarrow \mathcal{O}_{Y,y} \xrightarrow{s} \mathcal{O}_{Y,y}^{\oplus(r+1)} \longrightarrow \mathcal{E}_y \longrightarrow 0$$

with $s(1) = (t_1, \ldots, t_{r+1})$ where $(t_1, \ldots, t_{r+1})$ are regular generators of the maximal ideal of the local ring $\mathcal{O}_{Y,y}$. Bănică called such sheaves smooth and studied them in [B], see also [BW].

(3.5). According to (3.4) the study of Fano manifolds of middle index and $b_2 \geq 2$ can be split into three cases. We will do it below. The notation is consistent with (3.1)—(3.4) and $\mathcal{E}$ is always $p_*\mathcal{O}_X(H) = p_*\mathcal{O}_X(-K_X/r)$.

(4.1). First we present the case of projective bundles. The classification was done in joint paper with Thomas Peternell and Michał Szurek [PSW1]. The vector bundle $\mathcal{E}$ is then ample of rank $r = dimY - 1$ and $c_1\mathcal{E} = -K_Y$. The result is as follows.

No.	Base of p	projective bundle	"the other contraction"
1.	V_d	$\mathbf{P}(\mathcal{O}_V(1)^{\oplus r})$	$V \times \mathbf{P}^{r-1} \to \mathbf{P}^{r-1}$
2.	$\mathbf{P}^{r+1}$	$\mathbf{P}(\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus(r-2)})$	small
3.	$\mathbf{P}^{r+1}$	$\mathbf{P}(\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus(r-1)})$	divisorial
4.	$\mathbf{Q}^{r+1}$	$\mathbf{P}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)})$	divisorial
5.	$\mathbf{Q}^4$	$\mathbf{P}(\mathbf{E}(1) \oplus \mathcal{O}(1))$	scroll over $\mathbf{P}^3$

(4.2). Remarks. The variety V_d in the case 1 is a del Pezzo variety od degree d and $1 \leq d \leq 5$. The contraction φ is birational in cases 2—4, its exceptional set is a set of codimension 2 in case 2 or a divisor in cases 3 and 4, and the target is in these cases a cone. In case 5 the vector bundle $\mathbf{E}$ is spinor bundle over $\mathbf{Q}^4$ (see [O]), the contraction φ makes X a scroll with a 4-dimensional fiber, as described in [BSW, Example 3.2.4].

(4.3). It should be noted that apart from the above varieties one more possibility may a'priori occur. Namely, the following possibility was not excluded: X may have two $\mathbf{P}^2$-bundle structures over smooth Fano 4-folds Y_1 and Y_2 which are of index 1 and for any rational curve $C \subset Y_i$ there is $-K_{Y_i}.C \geq 3$.

This is because the proof of the classification of the above projective bundles is based on a "comparison lemma" which compares on $\mathbf{P}(E)$ three families of rational curves: these in the fibers of p and φ, and minimal sections over minimal rational curves in Y. The argument, based on comparing deformations of these, fails if these families are too small. This possibility does not seem to be probable. For details see [PSW1].

(5.1). A classification of Fano manifolds of middle index which have quadric bundle structure is done in [W5]. The Fano n-fold is then a divisor in $\mathbf{P}(\mathcal{E})$ of relative degree 2. More precisely, by adjunction

$$X \in |\mathcal{O}_{\mathbf{P}(\mathcal{E})}(2) \otimes \bar{p}^*\mathcal{O}_Y(-K_X - c_1\mathcal{E})|$$

where $\bar{p} : \mathbf{P}(\mathcal{E}) \to Y$ is the projection from the projective bundle. The result can be subsumed as follows

No.	Base of p	$\mathcal{E}$	other description
1.	$\mathbf{Q}^r$	$\mathcal{O}(1)^{r+2}$	$\mathbf{Q}^r \times \mathbf{Q}^r$
2.	$\mathbf{P}^r$	$\mathcal{O}(1)^{r+2}$	divisor $(1,2) \subset \mathbf{P}^r \times \mathbf{P}^{r+1}$
3.	$\mathbf{P}^r$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{r+1}$	blow-up of $\mathbf{Q}^{r-1} \subset \mathbf{Q}^{2r}$
4.	$\mathbf{P}^r$	$T\mathbf{P}^r \oplus \mathcal{O}(1)^2$	divisor $(1,1) \subset \mathbf{P}^r \times \mathbf{Q}^{r+1}$
5.	$\mathbf{P}^r$	$\mathcal{O} \oplus \mathcal{O}(1)^{r+1}$	double cover of $\mathbf{P}^r \times \mathbf{P}^r$

(5.2). Remarks. Above, in the third case, the quadric $\mathbf{Q}^{2r}$ is blown-up along its linear section $\mathbf{Q}^{r-1} = \mathbf{Q}^{2r} \cap \mathbf{P}^r$. The branch divisor in 5. is of bidegree $(2,2)$. Moreover, it is worthwile to note that the bundle $\mathcal{E}$ is not ample in this case.

(6.1). In a joint paper with Edoardo Ballico [BW] we study non-locally free sheaves whose projectivisation gives Fano manifolds of middle index. It is proved that the local extension enjoyed by the sheaves from (3.4) can be then extended to a global locally free sheaf $\mathcal{F}$ of rank $r+1$, that is, we have a sequence

(6.2) $$0 \longrightarrow \mathcal{O} \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{E} \otimes \mathcal{L} \longrightarrow 0$$

where $\mathcal{L}$ is a line bundle over Y. The above extension is not unique, nor is the choice of the line bundle $\mathcal{L}$. The section s of the locally free sheaf $\mathcal{F}$ vanishes exactly at $c_{r+1}(\mathcal{F})$ different points which correspond to singular fibers of the map p.

(6.3). The study of sheaves occuring in the case *(iii)* of (3.4) is similar to the one concerning locally free sheaves. Namely, we use a comparison lemma to study "the other" contraction of X and subsequently to prove that Y is either $\mathbf{P}^{r+1}$ or $\mathbf{Q}^{r+1}$. Then we extend, as in (6.2), $\mathcal{E}(-1)$ and $\mathcal{E}$ to a locally free sheaf which we call, respectively, $\mathcal{F}_1$ and $\mathcal{F}_0$. Since $\mathcal{E}(-1)$ is numerically effective, it follows that $\mathcal{F}_1$ is numerically efective as well and thus we may use results from [PSW2]. Consequenly we can realise X as a divisor in a projective bundle which is a Fano manifold. We obtain the following examples:

No.	Base of p	$\mathcal{F}_0$	$\mathcal{F}_1$	other description
1.	$\mathbf{Q}^{r+1}$	$\mathcal{O}(1)^{r+1}$	$\mathcal{O}^{r+2}/\mathcal{O}(-1)$	quadric bundle #4
2.	$\mathbf{P}^{r+1}$	$\mathcal{G}$	$\mathcal{O}^{r+2}/\mathcal{O}(-2)$	quadric bundle #2
3.	$\mathbf{P}^{r+1}$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^r$	$(T\mathbf{P}(-1) \oplus \mathcal{O}(1))/\mathcal{O}$	blow-up of $\mathbf{P}^{r-1} \subset \mathbf{Q}^{2r}$
4.	$\mathbf{P}^{r+1}$	$T\mathbf{P}^{r+1}$	$\mathcal{O}^{r+3}/\mathcal{O}(-1)^2$	$(1,1) \cap (1,1) \subset \mathbf{P}^{r+1} \times \mathbf{P}^{r+1}$

Note that 1, 3 and 4 are divisors in varieties from Theorem (2.2). All projective bundles $\mathbf{P}(\mathcal{F}_i)$ occurring in the table, except $\mathbf{P}(\mathcal{G})$, are Fano manifolds. $-K_{\mathbf{P}(\mathcal{G})} = \mathcal{O}_{\mathbf{P}(\mathcal{G})}(r)$ is not ample but it is nef; actually, $\mathcal{G}$ is spanned and the map associated to $\mathcal{O}_{\mathbf{P}(\mathcal{G})}(1)$ contracts to point a section of the projective bundle over a hyperplane in $\mathbf{P}^{r+1}$.

(7.1). The above classification shows that the structure of Fano manifolds of the middle index is richer in lower dimensions. For example, the variety $V_5 \times \mathbf{P}^{r-1}$ occurs only for $n \leq 10$ and there are two special 6-folds: the case 5 in (4.1) and $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$.

In dimension 4 the second example in the list of projective bundles does not occur. On the other hand we have the following list of Fano manifolds of index 2 which do not occur in higher dimensions:

$b_2 = 2$, $\mathbf{P}(\mathbf{N}(1)) = \mathbf{P}(\mathbf{E}(1))$, two $\mathbf{P}^1$ structures, where $\mathbf{N} = \Omega\mathbf{P}^3(2)/\mathcal{O}$ is a null-correlation bundle on $\mathbf{P}^3$ and $\mathbf{E}$ is a spinor bundle on $\mathbf{Q}^3$,
$b_2 = 2$, $\mathbf{P}^1 \times \mathbf{P}^3$,
$b_2 = 3$, $\mathbf{P}^1 \times \mathbf{P}(T\mathbf{P}^2)$, three $\mathbf{P}^1$-bundle structures,
$b_2 = 3$, $\mathbf{P}^1 \times \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2) \oplus \mathcal{O}_{\mathbf{P}^2}(1))$,
$b_2 = 4$, $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

(7.2). Note that we obtain the following

Corollary. *Let X be a Fano manifold of middle index and $b_2 = 2$. Then $H = -K_X/r$ is a sum of two numerically effective divisors which define elementary contractions of X.*

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