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MANIFOLDS AND CELL COMPLEXES

Fano 4-folds of Index 2 with $b_2 \geqslant 2$
A Contribution to Mukai Classification

by

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Summary. The paper provides a complete proof of a part of classification of Fano manifolds of coindex 3, announced by Mukai in 1982.

A smooth projective variety X of dimension n defined over the field of complex numbers is called a Fano n-fold if and only if its anticanonical divisor $-K_X$ is ample. The index of such a manifold is defined as the largest integer dividing $-K_X$, i.e.:

$$\operatorname{index}(X) = \max\{k \in \mathbb{Z} : -K_X \simeq kH \text{ for some ample divisor } H\}$$

In [13] Wilson proved that if X is a Fano 4-fold of index 2 with $b_2 = 1$ then the divisor H in the above definition can be assumed to be smooth. In the present paper we make the following:

ASSUMPTION 0.1. *X is a Fano 4-fold with second Betti number $b_2(X) \geqslant 2$ and on X there exists an ample smooth divisor H such that $2H$ is linearly equivalent to $-K_X$.*

We define the degree of a Fano manifold of index 2 as the selfintersection of H, i.e.:

$$d(X) = H^{\dim X} = \left(-\tfrac{1}{2}K_X\right)^{\dim X}$$

Let us recall that Iskovskich [1, 2] classified Fano 3-folds of index 2. There are 8 types (up to deformation) of them. They are as follows:

(i) V_d, $d = 1, \ldots, 5$ with $b_2(V_d) = 1, d(V_d) = d$ (see [1] for a thorough description);

(ii) $V =$ blow-up of P^3 at a point $= P(\mathcal{O}_{P^2}(-1) \oplus \mathcal{O}_{P^2})$, $b_2(V) = 2$, $d(V) = 7$;

(iii) $W =$ divisor of bidegree (1,1) on $P^2 \times P^2 = P(TP^2(-2))$, $b_2(W) = 2$, $d(W) = 6$;

(iv) $P^1 \times P^1 \times P^1$, $d(P^1 \times P^1 \times P^1) = 6$.

The purpose of this paper is to prove the following:

THEOREM 0.2: *Assume that X and H be as in* (0.1). *Then the pair (X, H) is one of the listed in Table* 0.3.

TABLE 0.3. Pairs (X, H)

No.	b_2	$d(X)$	X	H
1	2	4	$P^1 \times V_1$	No. 1
2	2	8	$P^1 \times V_2$	No. 3
3	2	12	$P^1 \times V_3$	No. 5
4	2	12	a double cover of $P^2 \times P^2$ whose branch locus is a divisor of bidegree (2,2)	No. 6b
5	2	16	a divisor on $P^2 \times P^3$ of bidegree (1,2)	No. 9
6	2	16	$P^1 \times V_4$	No. 10
7	2	20	an intersection of two divisors of bidegree (1,1) on $P^3 \times P^3$	No. 12
8	2	20	a divisor on $P^2 \times Q^3$ of bidegree (1,1)	No. 13
9	2	20	$P^1 \times V_5$	No. 14
10	2	22	a blow-up of Q^4 along a conic which is not contained in any plane lying on Q^4	No. 16
11	2	24	P(NCB), where NCB is the null-correlation bundle on P^3, (see (1.1.(iii)))	No. 17
12	2	26	a blow-up of Q^4 with center a line	No. 19
13	2	30	$P(\mathcal{O}_{Q^3}(-1) \oplus \mathcal{O}_{Q^3})$	No. 23
14	2	32	$P^1 \times P^3$	No. 25
15	2	40	$P(\mathcal{O}_{P^3}(-1) \oplus \mathcal{O}_{P^3}(1))$	No. 28
16	3	24	$P^1 \times W$	No. 7
17	3	28	$P^1 \times V$	No. 11
18	4	24	$P^1 \times P^1 \times P^1 \times P^1$	No. 1

The last column in Table 0.3 refers to entries in the table numbered as $b_2(X)$ in Mori-Mukai classification of Fano 3-folds (see [8]).

The proof of (0.2), suggested in the Mukai's paper [14], should be based on the classification of Fano 3-folds and Mori theory, which are also used extensively in the present paper. Moreover this paper applies results on Fano bundles [12] that were obtained without the assumption on smoothness of H. As, to my knowledge, no proof of Mukai classification has been published yet, therefore I have decided to come forward with the present paper as a contribution towards this classification.

1. Preliminaries; plan of the proof of (0.2). The proof of (0.2) will depend on results concerning Mori theory and classification of Fano 3-folds. We refer the reader to [6,7], for definitions concerning the cone of curves, extremal rays, contractions, etc. Our language and notation are consistent with these papers.

Using Mori theory and properties of vector bundles in [12], we classified all Fano 4-folds that are ruled, i.e., can be presented as P^1 bundles (Thm. (0.1) in [12]).

THEOREM 1.1. *Assume that X is a ruled Fano 4-fold of index 2. Then one of the following holds:*

(i) *$X = P^1 \times M$ where M is a Fano 3-fold of index 2 or P^3;*

(ii) *either $X = P(\mathcal{O}_{P^3}(-1) \oplus \mathcal{O}_{P^3}(1))$ or $X = P(\mathcal{O}_{Q^3}(-1) \oplus \mathcal{O}_{Q^3})$;*

(iii) *X has two P^1-bundle structures and can be realized either as $P(NCB)$, where NCB is the null-correlation bundle on P^3, that is a stable bundle with $c_1 = 0$ and $c_2 = 1$, or $P(E)$, where E is a stable rank-2 bundle on Q^3 with $c_1E = -1$, $c_2E = 1$.*

We proved also the following facts on maps from Fano 4-folds of index 2.

LEMMA 1.2. *Let $contr_R : X \to Y$ be a contraction of an extremal ray R of a Fano 4-fold X of index 2. If every fiber of $contr_R$ is of dimension $\leqslant 1$, then X is ruled.*

LEMMA 1.3. *Assume that $b_2(X) \geqslant 2$. If there exists an extremal ray R of X whose contraction has a 3-dimensional fiber then X is ruled.*

LEMMA 1.4. *Let X be a Fano 4-fold of index 2. If there exists a morphism from X onto a curve, then $X = P^1 \times M$ where M is either a Fano 3-fold of index 2 or P^3.*

Our plan for the proof of Theorem 0.2 is as follows: In Sections 2–5, we will assume that a pair (X, H) satisfies Assumption 0.1. In Section 2 we will prove that the system $|H|$ is almost always base point free and certain maps of H extend to X. In Section 3 using an extension lemma and Mori theory we will deal with the case of $b_2(X) \geqslant 3$. We will proof that such X has to be ruled, so it has to be one listed in (1.1). In Section 4 we will be interested in a case of $b_2(X) = 2$ and one of extremal rays of X being not effective. We will prove then that, X is a blow-up of Q^4 with center a smooth curve, and such a curve is either a line or a conic not contained in a plane on Q^4. In Section 5 we will assume that $b_2(X) = 2$ and both extremal rays are numerically effective, but X has no morphism onto P^1 (cf. Lemma 1.4). Using the classification of Fano 3-folds, [8], we will find all possible candidates for the divisor H, and then examine the structure X. The following fact will be used in Section 5.

LEMMA 1.5. *Let X be a Fano 4-folds of index 2 and H an ample divisor on X such that $-K_X \equiv 2H$ (H does not have to be smooth). Then, for $m \geqslant 0$*

$$h^0(X, \mathcal{O}_X(mH)) = (1/24)((m+1)^4H^4 + (m+1)^2(24 - H^4))$$

Proof. Since, for $m \geqslant 0$, $i > 0$, $H^i(X, \mathcal{O}_X(mH)) = 0$ then we are to prove that the Euler-Poincare characteristic of $\mathcal{O}_X(mH)$ can be expressed by the above polynomial. If we take $\chi(m) = \chi(X, \mathcal{O}_X(mH))$, then the following hold:

(a) $\chi(m)$ is a polynomial of degree 4 in m with the leading coefficient $(1/24)H^4$;

(b) $\chi(0) = 1$;

(c) $\chi(-1) = 0$, since by Kodaira vanishing theorem $h^i(X, \mathcal{O}_X(-H)) = 0$, $i > 0$;

(d) $\chi(m) = \chi(-m-2)$ (by Serre duality), which means that the polynomial $\chi(m+1)$ is even.

Now from (a) and (d) it follows that $\chi(m) = (1/24)H^4(m+1)^4 + B(m+1)^2 + C$ where B and C are rationals. Using property (b) and (c) we get that $C = 0$ and $B = 1 - (1/24)H^4$.

2. Extension lemmas. Let X and H be as in Assumption 0.1. By $N_1(X)$ ($N_1(H)$) and $NE(X)$ ($NE(H)$) let us denote the space of 1-cycles and the cone of effective 1-cycles on X (on H respectively), cf. [7]. In view of the Lefschetz hyperplane section theorem we see that the embedding $H \subset X$ gives us isomorphism $N_1(H) \simeq N_1(X)$ under which $NE(H) \subset NE(X)$.

LEMMA 2.1. $NE(H) = NE(X)$.

Proof. Since $NE(X)$ is spanned on its extremal rays, it follows that the lemma is proved if we show that any contraction $\mathrm{contr}_R : X \to M$ of an extremal ray R of X contracts a curve lying on H. This is obvious if contr_R has a 2-dimensional fiber, since then H has a positive-dimensional intersection with this fiber. On the other hand if contr_R has no fiber of dimension $\geqslant 2$, then M is smooth and $\mathrm{contr}_R : X \to M$ is a P^1 bundle (cf. Lemma 1.2.). In this case we see that H (fiber of contr_R) $= 1$, hence the map $\mathrm{contr}_{R|H} : H \to M$ is birational. If $\mathrm{contr}_{R|H}$ had no positive-dimensional fiber then H would be a section of π which is an absurd since $b_2(X) = b_2(H) > b_2(M)$.

Corollary 2.2. *Any ample (nef) line bundle on H extends to ample (respectively nef) line bundle on X.*

Lemma 2.3. *Let $\varphi_H : H \to Y$ be a morphism onto a projective variety Y. Assume that one of the following holds:*
(i) *$dimY \leqslant 2$;*
(ii) *φ_H is a contraction of an extremal ray R_H in $NE(H)$ and this ray treated as ray R_X in $NE(X)$ is numerically effective.*
Then φ_H extends to $\varphi : X \to Y$.

Proof. We follow Sommese's ideas from [10]. Let $L_H = \varphi_H^*(L_Y)$, where L_Y is a very ample line bundle on Y. The line bundle L_H extends to a nef line bundle L_X on X. First we prove that every section of L_H extends to a section of L_X. To see this let us consider a short exact sequence on X:

$$0 \to L_X \otimes \mathcal{O}(-H) \to L_X \to L_H \to 0 .$$

Any section of L_H extends to a section of L_X if $H^1(X, L_X \otimes \mathcal{O}(-H)) = 0$. However, L_X is nef and therefore $L_X \otimes \mathcal{O}(H) = L_X \otimes \mathcal{O}(-H) \otimes \mathcal{O}(-K_X)$ is an ample line bundle on X. Now the desired vanishing follows from Kodaira vanishing theorem.

Let $\mathrm{Bs}(L_X)$ denote the base point set of $|L_X|$. Since $|L_X|$ has no base points on H, it follows that $\dim \mathrm{Bs}(L_X) \leqslant 0$. We claim that actually the set $\mathrm{Bs}(L_X)$ is empty. To see this note that L_X is semi-ample because of Kawamata-Shokurow base point free theorem, 3–1–1 [6], therefore for some $m \gg 0$ the system $|mL_X|$ is base point free and defines a map $\Phi_{|mL_X|} : X \to P^{\dim |mL_X|}$. Now either of the assumptions, (i) or (ii), implies that all fibres of $\Phi_{|mL_X|}$ are of positive dimension. Therefore either $\mathrm{Bs}(L_X)$ is empty or of positive dimension, and the latter case we excluded above.

Finally we see that $\varphi = \Phi_{|L_X|}$, the map associated to $|L_X|$, is the desired extension of φ_H.

In view of Lemma 1.4. we have:

Corollary 2.4. *If H has a morphism onto a curve then $X = P^1 \times M$ where M is either a Fano 3-fold of index 2 or P^3.*

Corollary 2.5. *The divisor H can not be represented as a nontrivial product.*

We conclude this section with:

Lemma 2.6. *The linear system $|H|$ is base point free unless the pair (X, H) is the one listed as No. 1 in Table 0.3.*

Proof. We see that the line bundle $\mathcal{O}_X(H)$ is not spanned only if $\mathcal{O}_H(H)$ is not spanned. Thus we are looking for a Fano 3-fold H index 1 such that $|-K_H|$ is not base point free. This happens only if $H = P^1 \times S_1$ (where S_1 is a del-Pezzo surface, such that $K^2_{S_1} = 1$) or H is the manifold listed as n^0 1 in table 2, [8]. The first case is ruled out by Corollary 2.5. In the other case H has a morphism onto P^1, hence X has to be a product Lemma 2.4.), and by inspection we conclude that the pair (X, H) is No. 1 in table 0.3.

3. Case of $b_2(X) \geqslant 3$. Assume that the pair (X, H) is as in 0.1. and $b_2(X) = b_2(H) \geqslant 3$. We will prove that X is ruled. For that purpose, in view of Corollary 2.4., we may assume that H has no morphism onto P^1.

From table 6 in [8] we infer that in this case $b_2(H) = 3$ and there exists a morphism $\varphi_H : H \to P^2$ which makes H a conic bundle. In virtue of Lemma 2.3. the morphism φ_H extends to $\varphi : X \to P^2$. We see that all fibers of φ are connected, of dimension 2. Let $L = \varphi^*(\mathcal{O}_{P^2}(1))$. Then, in terminology of [6] φ is a contraction of an extremal face $\Sigma = \{Z \in N_1(X) : L \cdot Z = 0\} \cap NE(X)$ of the cone $NE(X)$. The face Σ is of dimension 2 and contains 2 extremal rays. For any of these extremal rays there exist a projective normal variety Y, morphism of contraction $\Phi : X \to Y$ and a morphism $\sigma : Y \to P^2$ which make the following diagram commute (cf. Thm. 3.2.1 [ibid]):

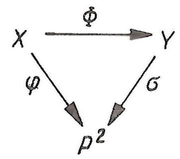

Fig. 1.

Now the task of this section is achieved in the following:

Lemma 3.1. *$\Phi : X \to Y$ is a P^1-bundle.*

Proof. As in Lemma 1.2. it is enough to prove that Φ has no 2-dimensional fibers. Note that fibers of Φ are contained in fibers of φ that are of dimension 2. Let f denote a fiber of φ. We claim that if f contains a 2-dimensional fiber of Φ then Φ must actually contract f to a point. It is clear when f is irreducible. If f is reducible then, since $H \cap f$ is a conic and $\mathcal{O}_f(H)$ is ample and spanned, it follows (Thm. 2.1 b′ [4]) that f must consist of two copies of P^2 intersecting along 1-dimensional set. Therefore if Φ contracts one component of f it must contract whole f. This implies that there exists a fiber of σ that is a point hence Y is of dimension 2. Furtheremore, the map σ is finite-to-one and since fibers of both φ and Φ are connected, σ has to be an isomorphism. It would imply that φ and Φ are the same (modulo σ) which is an absurd since φ contracts 2-dimensional face and Φ is a contraction of an extremal ray.

4. Case of $b_2(X) = 2$ and an extremal ray which is not numerically effective. Assume that the pair (X, H) is as in Assumption 0.1 and moreover $b_2(X) = b_2(H) = 2$ and one of (two) extremal rays of X is not numerically effective. We will prove that (X, H) is one of No. 10, 12, 13, 15 from Table 0.3.

Let R denote an extremal ray of X which is not numerically effective. Let $E \subset X$ be the prime divisor from lemma 1.1. [12], such that $E \cdot R < 0$. The contraction $\varphi : X \to Y$ of R is a birational morphism onto a normal projective variety Y, and E is the exceptional set of φ. By virtue of Lemma 1.1 [12], it follows that $\dim \varphi(E) \leqslant 1$. If $\dim \varphi(E) = 0$ then from Lemma 1.3. it follows that X is ruled and we see that actually (X, H) is either No. 13 or 15 from Table 0.3. Therefore, for the rest of this section, we may assume that $\dim \varphi(E) = 1$. If we take $C = \varphi(E)$ (with reduced structure), then C is an irreducible curve on Y. Moreover we have:

LEMMA 4.1. *Both Y and C are smooth and $\varphi : X \to Y$ is a blow-up of Y along C.*

Proof. In view of [9] it is enough to prove that every fiber of $\varphi_{|E} : E \to C$ is isomorphic to P^2 and $\mathcal{O}_X(E)$ restricted to any fiber of $\varphi_{|E}$ is isomorphic to $\mathcal{O}_{P^2}(-1)$.

We start by restricting the map φ to H. If H_Y denotes the image $\varphi(H)$ then the map $\varphi_{|H} : H \to H_Y$ is birational. We claim that H is a blow-up of H_Y along C. To see it let us take $\nu : Z \to H_Y$ as a normalization of H_Y, and $\alpha : H \to Z$ as a morphism making the following diagram commutative:

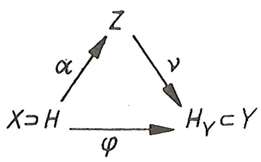

Fig. 2.

The morphism $\alpha : H \to Z$ is the contraction of the extremal ray R on H and from (3.3.1, [7]) we see that Z is smooth and H is a blow-up of Z along a smooth curve C_Z such that $(C_Z) = C$. From ([8], Table 6) we see that Z is actually one of the following: P^3, Q^3 or V_d, for $3 \leqslant d \leqslant 5$. Now we are to prove that ν is an isomorphism. Let L_Y be a very ample line bundle on Y and $L = \varphi^*(L_Y)$. As in the proof of Lemma 2.3 we see that any section of $L_H := L_{|H}$ extends to X. Therefore the morphism $\varphi_{|H}$ is given by the linear system $|L_H|$. But at the same time L_H is a pullback of a very ample line bundle from Z (because every ample line bundle on $Z = P^3$, Q^3 or V_d, $d = 3, \ldots, 5$, is very ample) hence ν is an isomorphism. Thus $\varphi_{|H} : H \to H_Y$ is a blow-up along smooth C.

The exceptional set of $\varphi_{|H}$ equals to $H_E = H \cap E$ which is P^1-bundle over C. If f is a fiber of $\varphi_{|H_E} : H_E \to C$ then from adjunction it follows that $H \cdot f = 1$ and $E \cdot f = -1$. The map $\varphi_{|E} : E \to C$ is flat and if F is a fiber of $\varphi_{|E}$, then the selfintersection of $\mathcal{O}_F(H)$ is 1. Now from (Thm. 2.1.b$'$, [4]) it follows that any fiber F of $\varphi_{|E}$ is isomorphic to P^2 and $\mathcal{O}_F(E) \simeq \mathcal{O}_{P^2}(-1)$ which concludes the proof of Lemma 4.1.

Now we see that Y is isomorphic to Q^4. Indeed, $\operatorname{Pic} Y = Z$ hence H_Y is an ample divisor and, as we noted above, a Fano 3-fold of index $\geqslant 2$. Therefore Y is Fano of index $\geqslant 3$, and since $-K_X = -\varphi^* K_Y - 2E$ has to be divisible by 2, it has to be actually of index 4, hence a quadric.

Finally, we see that H_Y is isomorphic to V_4 and since H is a blow-up of H_Y along C it has to be isomorphic to one of 3-folds numbered as 10, 16 or 19 in (Table 2 [8]). We eliminate No. 10 since it has a morphism onto P^1. The description of C follows from the description of H in (Table 2 [8]).

5. $b_2(X) = 2$ and both extremal rays are nef. In this section we are working in the following set-up: the pair (X, H) is, as in (0.1), $b_2(X) = 2$ and both extremal rays of X are numerically effective. This means that any contraction $\varphi : X \to Y$ is onto a normal variety Y of dimension $\leqslant 3$. Moreover, in view of Lemmas 1.3 and 1.4 we assume that no contraction of X onto a curve (i.e. $\dim Y \geqslant 2$) and no contraction of X has a fiber of dimension 3.

We start by examining the structure of H. For this purpose we will frequently refer to (Table 2 in [8]) and, unless otherwise specified, we will use ordinals from this table to describe isomorphism classes of H. For example, H is not isomorphic to any of 3-folds numbered as 1, 2, 3, 4, 5, 7, 10, 14, 18, 25, 29, 33, 34, since these have morphisms onto P^1, (cf. Corollary 2.4). On the other hand, by virtue of (Thm. 2.0, [3]), H can not be a P^1-bundle over P^2, therefore it is not isomorphic to any of 3-folds numbered as 24, 27, 31, 32, 34, 35, 36 (cf. [11]).

Let $\varphi_H^1 : H \to Y_1$, $\varphi_H^2 : H \to Y_2$ be the two contractions of H, $2 \leqslant \dim Y_i \leqslant 3$ for $1 \leqslant i \leqslant 2$. From Lemma 2.3 it follows that any of φ_H^i extends to $\varphi_i : X \to Y_i$, and we see that these are the two contractions of X. We have:

LEMMA 5.1. *If* $\dim Y_i = 3$ *then there exists at most a finite number of fibers of* φ_i *of dimension* 2, *every of them isomorphic to* P^2.

Proof. Let $D_i = \varphi_i^*(D_{Y_i})$ where D_{Y_i} is a very ample divisor on Y_i. If there is more than a finite number of φ_i fibers of dimension 2 then φ_i contracts a divisor, say E, to a curve. Since E does not meet other fibers of φ_i and $b_2(X) = 2$, it follows that E has to be numerically equivalent to some multiple of D_i. However, this cannot be true since $D_i^2 \cdot E \equiv 0$ and on the other hand D_i^3 is an effective 1-cycle. This concludes the first part of the lemma.

Now take S to be a 2-dimensional fiber of φ_i. We claim that for a general H the intersection $f = H \cap S$ is isomorphic to P^1 and $\mathcal{O}_f(H) \simeq \mathcal{O}_{P^1}(1)$. Indeed, since the number of 2-dimensional fibers of φ_i is finite, it follows that, for a general H, the map $\varphi_{i|H} : H \to Y_i$ has no 2-dimensional fiber, therefore by (3.3.1, [7]), it is a blow-down map. Now f, being a 1-dimensional fiber of $\varphi_{i|H}$, is isomorphic to P^1 and $\mathcal{O}_f(H) \simeq \mathcal{O}_{P^1}(1)$. Now, we see that the pair $(S, \mathcal{O}_S(H))$ satisfies the assumption b′) of (Thm. 2.1, [4]), therefore $S \simeq P^2$. □

Note that from the proof of Lemma 5.1 it follows that we can assume that no contraction of H has a 2-dimensional fiber, therefore H is not isomorphic to any of 3-folds listed as numbers 8, 15, 23, 28, 30. Moreover by virtue of (Thms 3.3 and 3.5 from [7]) it follows that both Y_i are smooth.

LEMMA 5.2. *If* $\dim Y_2 = 3$ *and* X *is not ruled then* Y_1 *is isomorphic either to* P^2 *or* P^3.

Proof. In view of Lemmas 1.2 and 5.1 it follows that there exists a fiber S of φ_2 isomorphic to P^2. We claim that $\varphi_{1|S}$ is an embedding defined by the system $|\mathcal{O}_{P^2}(1)|$. Indeed, $\mathcal{O}_S(H) \simeq \mathcal{O}_{P^2}(1)$, therefore, by classification of Fano 3-folds and their contractions (cf. Corollary 11.2 and Table 2 in [8]), it follows that the map $\varphi_{1|S}$ is given by the linear system $|\mathcal{O}_{P^2}(1)|$. This can be the case only if Y_1 is either P^2 or P^3.

Similarly we prove:

LEMMA 5.3. *If* $\dim Y_1 = 2$ *and* $\dim Y_2 = 3$ *then* Y_2 *is either* P^3 *or* Q^3.

Proof. Note that a general fiber of φ_1 is a smooth 2-dimensional quadratic $P^1 \times P^1$. Moreover $\mathcal{O}_X(H)$, restricted to this fiber, is isomorphic to $\mathcal{O}_{P^1 \times P^1}(1, 1)$. Now one concludes Lemma 5.3. as the proof of Lemma 5.2.

Let us consider a map $\Phi := \varphi_1 \times \varphi_2 : X \to Y := Y_1 \times Y_2$. We see that the map Φ is finite-to-one and if $\dim Y \geqslant 5$ $\Phi_{|H}$ is an embedding (cf. Table 2 [8]). Therefore for dim $Y \geqslant 5$ the map Φ is birational onto its image.

Now we are ready to describe the pair (X, H). From Table 2 [8] we see that $\dim Y = 4$ only if $Y_1 \simeq Y_2 \simeq P^2$ and H is in the class No. 6b. Therefore, the pair (X, H) is then the one listed as No. 4 in Table 0.3. If $\dim Y = 5$ then from Lemma 5.3. it follows that Y is isomorphic either to $P^2 \times P^3$ or $P^2 \times Q^3$. Moreover, we see that the map Φ is birational onto a divisor of bidegree (1,2) or (1,1), respectively. We claim that Φ is actually an embedding.

To see this set $Z = \Phi(X)$. Then Z is a divisor on Y and from the exact sequence

$$0 \to \mathcal{O}_Y(m,m) \otimes \mathcal{O}_Y(-Z) \to \mathcal{O}_Y(m,m) \to \mathcal{O}_Z(m,m) \to 0$$

one finds out that for $m \geqslant 0$

$$h^0(Z, \mathcal{O}_Z(m,m)) = h^0(X, \mathcal{O}_X(mH))$$

(cf. Lemma 1.5). Now our claim follows by:

LEMMA 5.4. *Let H be an ample divisor on a manifold X. Assume that $|H|$ is base point free and the map $\Phi : X \to P^{\dim |H|}$ is onto a variety $Z \subset P^{\dim |H|}$. By $\mathcal{O}_Z(m)$ let us denote the restriction of $\mathcal{O}_{P^{\dim |H|}}(m)$ to Z. If, for any $m \geqslant 0$*

$$h^0(X, \mathcal{O}_X(mH)) = h^0(Z, \mathcal{O}_Z(m))$$

then Φ is an embedding.

Proof. For some $m > 0$ the map $\Phi_m : X \to P^N$ associated to a complete linear system $|mH|$ is an embedding. But we see that $|mH| = \Phi^*|\mathcal{O}_Z(m)|$, therefore, if $\Phi_m^Z : X \to P^N$ is the map associated to $|\mathcal{O}_Z(m)|$, then the following diagram commutes:

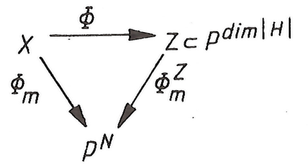

Fig. 3.

Thus Φ is an isomorphism onto Z.

Now let us deal with the case of $\dim Y = 6$. In view of Lemma 5.2. it follows that either X is ruled, which is the case of No. 11 in Table 0.3, or $Y_1 \simeq Y_2 \simeq P^3$. In the latter case H is in class No. 12 (Table 2, [8]) or, in other words, it is a graph of a cubo-cubic Cremona transformation [5]. Therefore $d(X) = 20$ and by Lemma 1.5 we see that $h^0(X, \mathcal{O}_X(H)) = 14$. Consider $Z \subset P^3 \times P^3$ being the image of Φ. We claim that Z is a complete intersection of two divisors of bidegree (1.1).

To see this note that:

$$h^0(P^3 \times P^3, \mathcal{O}_{P^3 \times P^3}(1,1)) = 16$$

therefore, there exists a linear pencil in $|\mathcal{O}_{P^3 \times P^3}(1,1)|$ of divisors containing Z. Note that no divisor in this pencil is reducible, since if it was $\Phi(X)$, contained in its component, it would have a morphism onto P^2. Let $\overline{Z}$ be the common zero set of this pencil. Then $\overline{Z}$ is an algebraic variety of pure dimension 4 and Z is its irreducible component. But note that:

$$\overline{Z} \cdot (1,1)^4 = (1,1)^6 = 20 = d(X) = \Phi(X) \cdot (1,1)^4 = Z \cdot (1,1)^4$$

therefore $\overline{Z} = Z$.

Now, using similar argument as before, we find out that for any $m \geqslant 0$

$$h^0(X, \mathcal{O}_X(mH)) = h^0(Z, \mathcal{O}_Z(m,m))$$

therefore from Lemma 5.4 it follows that $X \simeq Z$, hence the pair (X, M) is as No. 7 in Table 0.3.

This concludes the proof of Theorem 0.2.

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