Ample vector bundles and introduction to Mori theory

Fall 2014

Problem set V, for November 13th

All varieties are defined over an algebraically closed field k.
Recall the following basic notions which can be found in the Hartshorne's book.
Differentials. Let A be a finitely generated k algebra (assume it is a domain). The A module Ω_A is generated by symbols da, with $a \in A$ modulo the relations: (1) $d(a+b) = da + db$, (2) $d(ab) = adb + bda$, (3) $d_{|k} = 0$. On a variety X we define the coherent sheaf Ω_X by glueing its values on the affine pieces.
Coherent sheaves on the projective space. Let $S = \bigoplus_{d \geq 0} S^d = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n = \operatorname{Proj} S$ which is covered by affine sets $U_i \simeq \mathbb{A}^n_k$ with coordinates $k[x_o/x_i, \ldots, x_n/x_i]$. Recall that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ we associate a graded S-module of its sections $\Gamma^*(\mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$. On the other hand, for a graded S-module $M = \bigoplus_{j \in \mathbb{Z}} M^j$ we define a quasi-coherent sheaf $\widetilde{M}$ on $\mathbb{P}^n$ such that on every affine set U_i we have $\widetilde{M}(U_i) = M^0_{x_i} = \{m/x_i^r : m \in M^r\}$.

1. Euler sequence [Hartshorne, II.8]. In the notation introduced above, consider a morphism of graded S modules

$$\bigoplus_{j=0}^{n} S(-1) \cdot v_j \longrightarrow S$$

where v_j's are generators of degree 1 and each v_j is mapped to x_j. Let M be the kernel of this map.

(a) Show that we get the exact sequence of $\mathcal{O}$ sheaves

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$

(b) On U_i with coordinate ring $k[x_0/x_i, \ldots, x_n/x_i]$ we define a map $\psi_i : \Omega_{U_i} \to \mathcal{O}(-1)^{\oplus n+1}_{|U_i}$ by setting $\psi_i(d(x_r/x_i)) = (x_i v_r - x_r v_i)/x_i^2$. Prove that the map is isomorphism onto $\widetilde{M}_{|U_i}$.

(c) Prove that the homomorphisms ψ_i glue to isomorphism of sheaves $\Omega_{\mathbb{P}^n} \to \widetilde{M}$ so that we have the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$

2. Find the connecting (boundary) homomorphism $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}) \to \mathrm{H}^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n})$ in the sequence of cohomology of the Euler sequence; calculate it using the covering (U_i) and express it in terms of Čech cohomology.

3. For $d > 0$ consider the twisted Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \otimes \mathcal{O}(d) \longrightarrow \mathcal{O}(d-1)^{\oplus n+1} \longrightarrow \mathcal{O}(d) \longrightarrow 0$$

Prove that the H^0 of this sequence is exact and that it actually splits by $\mathcal{O}(d) \ni s \to \sum(\partial s/\partial x_i) \cdot v_i \in \mathcal{O}(d-1)^{\oplus(n+1)}$.

4. Use Euler sequence to calculate cohomology of sheaves $\Omega^p_{\mathbb{P}^n}$ of p-forms on $\mathbb{P}^n$. Hint: prove the following lemma. Given an exact sequence of vector bundles $0 \to V \to W \to L \to 0$, with L of rank 1, for any positive p we get the exact sequence $0 \to \Lambda^p V \to \Lambda^p W \to \Lambda^{p-1} V \otimes L \to 0$.

5. Prove the relative version of the Euler sequence: Let E be a vector bundle over smooth variety X. Consider its projectivisation $p : \mathbb{P}(E) \to X$. By $\Omega_{\mathbb{P}(E)/X}$ we denote the sheaf of relative differentials which are in the kernel of the map $Dp : p^*\Omega_X \to \Omega_{\mathbb{P}(E)}$. Then we have the following relative Euler sequence on $\mathbb{P}(E)$:

$$0 \longrightarrow \Omega_{\mathbb{P}(E)/X} \longrightarrow p^*E \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

Use the relative Euler sequence to calculate the canonical divisor of $\mathbb{P}(E)$ in terms of K_X, $detE$, rkE and $\mathcal{O}_{\mathbb{P}(E)}(1)$.