$\mathbb{C}^*$ quotients of varieties with big torus action

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acknowledgments, relation to previous research

- Work in progress[†] with Marysia Donten-Bury, based on previous joined projects with Jarek Buczyński, Andrzej Weber, Mateusz Michałek, Leonid Monin, Gianluca Occhetta, Eleonora Romano, and Luis Solá Conde
- Kapranov, Sturmfels, Zhelevinsky, Quotients of toric varieties, Math. Ann. 1999
- Bäker, Hausen, Keicher, On Chow quotients of torus actions, Mich. Math. 2015
- Goresky, Kottwitz, MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, Inv. Math. 1998

† so the talk will be abut ideas not results

torus in action, local linearization

- Let $H = (\mathbb{C}^*)^r$ denote an algebraic torus with $M = \mathbb{Z}^r$ the lattice of characters; we usually assume $r \geq 2$
- Assume that H acts faithfully on a normal variety X, with $\dim X = n$, we will usually assume that X is smooth and projective.
- We have decomposition of the set of fixed points into connected components

$$X^H = Q_1 \sqcup \cdots \sqcup Q_s$$

- For $p \in X^H$ the action of H on T_pX is linear and (by local linearization) up to étale cover can be identified with the action of H on X in a neighborhood of p.

torus in action, quotients

- Suppose that L is a line bundle over X (we usually assume L ample).
- Given a linearization $\mu : H \times L \to L$ we have a decomposition of space of sections into eigenspaces

$$\mathrm{H}^0(X, L) = \bigoplus_{u \in M} \mathrm{H}^0(X, L)_u$$

- For a suitable u's we define GIT quotients

$$Y_u = \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{H}^0(X, L^{\otimes m})_{mu}\right)$$

- Chow/Hilbert quotient is defined by the closure of the set of general orbits of the action in Chow/Hilbert scheme. Normalized Chow quotient is universal for all GIT quotients.

GKM action

- We say that the action of H on X is k-GKM if the number of orbits of dimension $\leq k$ is finite. In particular 0-GKM is finite number of fixed points, 1-GKM is just GKM, n-GKM is toric (here we do not assume that X is smooth).
- We say that the action is weak GKM if all $p \in X^H$ are isolated and the weights of the action of H on T_pH are $-\nu_1, \ldots, -\nu_n \in M$ and $\nu_i \neq \nu_j$ for $i \neq j$. The compass $\mathcal{C}_p$ is the set of ν_i's.
- If the action is weak GKM then all eigenspaces of the action in T_pX are of dimension 1 and (via local linearization) give rise to H-orbits, the closures of which we will call eigencurves (note that they are smooth).
- The action is GKM if and only if it is weak GKM and no two ν_i's are proportional.

context: Atiyah-Bott localization theorem

Suppose that the action of H on X is weak GKM and $X^H = \{y_1, y_2, \ldots, y_k\}$. For each fixed point y_i we consider its compass $\mathcal{C}_{y_i} = \{\nu_{i,j}, j = 1, \ldots, n\}$. For a line bundle L with a linearization μ by $\mu_i = \mu_L(y_i) \in M$ we denote the weight of the action of H on L_{y_i}.

Then the equivariant Euler characteristics

$$\chi^H(X, L) = \sum_{u \in M} \chi(X, L)_u t^u$$

can be computed in terms of μ_i's and $\nu_{i,j}$'s

$$\chi^H(X, L) = \sum_{i=1}^{k} \frac{t^{\mu_i}}{\prod_j (1 - t^{\nu_{i,j}})}.$$

motivation: understanding a birational map

Given $\mathbb{C}^*$ action on X with two GIT geometric quotients Y_1 and Y_2 we have a birational map $Y_1 \dashleftarrow\dashrightarrow Y_2$ and normalized Chow quotient $\mathcal{Y}$ resolves this map

$$\begin{array}{ccc} & \mathcal{Y} & \\ \swarrow & & \searrow \\ Y_1 & \dashleftarrow\dashrightarrow & Y_2 \end{array}$$

Example: inversion of (symmetric) matrices comes from a $\mathbb{C}^*$ quotients of (Lagrangian) Grassmanians and is resolved by complete collineations (quadrics) being Chow quotient of this action.

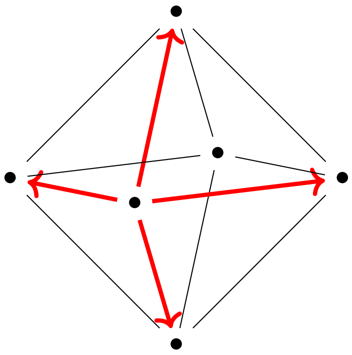
Problem Suppose X is GKM, describe GKM-like structure of $\mathcal{Y}$.

visualisation: a grid

A grid of GKM action of H on (X, L) is the set of $\mu_i = \mu_L(y_i)$ as points in M (as affine lattice), together with $\nu_{i,j}$'s as vectors (in the tangent lattice M) attached to μ_i's.

Take $\mathbb{Q}^4 = \{x_1x_2 + x_3x_4 + x_5x_6 = 0\} \subset \mathbb{P}^5$ with $(\mathbb{C}^*)^3$ action

$$(t_1, t_2, t_3) \cdot (x_1, \ldots, x_6) = (t_1x_1, t_1^{-1}x_2, \ldots, t_3x_5, t_3^{-1}x_6)$$

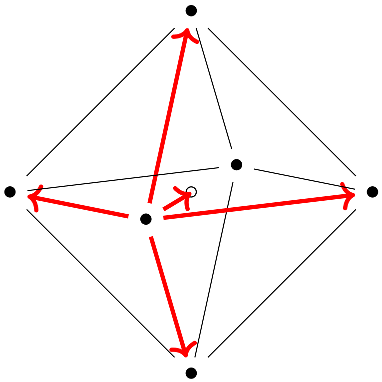


• denotes both fixed points and sections

grids of homogeneous varieties

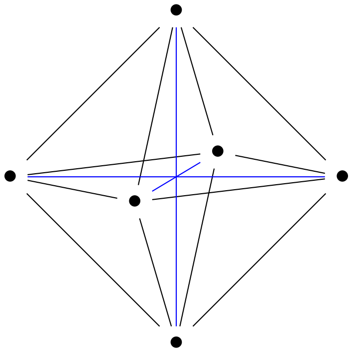
Given G simple with $P \subset G$ parabolic, the action of the Cartan torus H on G/P is GKM.
Take $\mathbb{Q}^5 = \{x_0^2 + x_1 x_2 + x_3 x_4 + x_5 x_6 = 0\} \subset \mathbb{P}^6$ with $(\mathbb{C}^*)^3$ action

$$(t_1, t_2, t_3) \cdot (x_0, x_1, \ldots, x_6) = (x_0, t_1 x_1, t_1^{-1} x_2, \ldots, t_3 x_5, t_3^{-1} x_6)$$



eigencurves, GKM graph

Instead of weights of eigenvectors (compasses) we can draw eigencurves, the result is GKM graph of $\mathbb{Q}^5$



The degree of an eigencurve with respect to L can be read from the grid via AMvsFM formula: if ν_{y_0} is a weight defining C then

$$\mu_L(y_0) + deg_C L \cdot \nu_{y_0} = \mu_L(y_\infty)$$

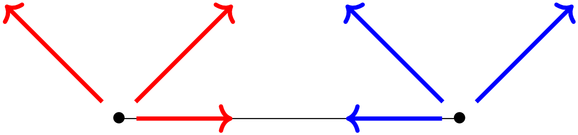
grid of a vector bundle over $\mathbb{P}^1$, downgrading

For $\mathbb{P}^1 \supset \mathbb{C}^*$ with (nonhomogeneous) coordinate z consider line bundle $\mathcal{O}(a)$ with local coordinates t_0, t_∞ satisfying relation

$$z^a \cdot t_0 = t_\infty$$

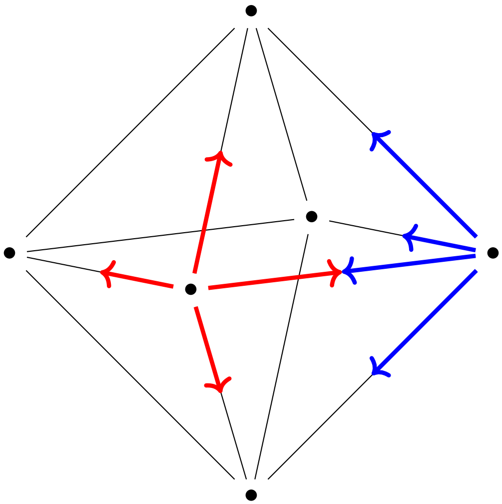
The rank 2 torus acting with weights $(1,0),(m,1)$ on (z,t_0) acts with weights $(-1,0),(m+a,1)$ on (z^{-1},t_∞)

The following grid of a rank 2 torus action (with $L = \mathcal{O}(3)$)



may present both $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$

example: normal to a line on $\mathbb{Q}^4$



note natural bijection between the vectors in compasses

strong GKM

A GKM action of H on X is strong GKM if for every fixed point $y \in X^H$ and every $\nu \in \mathcal{C}_y$ the quotient $M_\nu = M/\mathbb{Z}\nu$ has no torsion and the quotient map $M \to M_\nu$ is injective on $\mathcal{C}_y \subset M$.

Lemma Suppose that the action is strong GKM. Let C be an eigencurve with $C \cap X^H = \{p, p'\}$ and $-\nu, -\nu'$ respective weights whose eigenspaces are tangent to C in p, p', respectively.
Then $\nu' = -\nu$ and the isotropy of a general point of C is the torus $H_\nu = Hom(M_\nu, \mathbb{C}^*)$.
Moreover, up to renumbering, the remaining elements of compasses $\mathcal{C}_p = \{\nu, \nu_1, \ldots, \nu_{n-1}\}$ and $\mathcal{C}_{p'} = \{\nu', \nu'_1, \ldots, \nu'_{n-1}\}$ satisfy $\nu'_i = \nu_i - a_i\nu$ where a_i come from the (H equivariant) splitting of the normal $N_{C/X} = \bigoplus_i \mathcal{O}(a_i)$.

changing local coordinates along an eigencurve

In the situation of the previous slide we define a matrix

$$A_C = \begin{pmatrix} -1 & -a_1 & \cdots & -a_{n-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Let $\widetilde{M} = \mathbb{Z}^n$ with basis $v_0, v_1, \ldots v_{n-1}$ and the homomorphism $\widetilde{M} \to M$ sending $v_0 \mapsto \nu,\ v_1 \mapsto \nu_1, \cdots v_n \mapsto \nu_n$. If for $i = 0, \ldots, n-1$ we set $v_i' = A_C \cdot v_i$ then $v_0' \mapsto \nu'$ and $v_i' \mapsto \nu_i'$. That is A_C preserves grading† $\widetilde{M} \to M$; we call it a shift along C.

Note: Let $\mathbb{C}[\widetilde{M}] = \mathbb{C}[t^u : u \in \widetilde{M}]$ be a ring of Laurent polynomials with grading in M. Then its subrings $\mathbb{C}[t^{v_i}]$ and $\mathbb{C}[t^{v_i'}]$ represent coordinate rings of H equivariant étale neighborhoods of p, p'.

† it does not mean that $v_i \mapsto v_i'$ descends to automorphism of M

digression: case of homogeneous varieties

- For a simple group $G \supset H$ take Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$. For every root $\alpha \in R$ we have a copy of $\mathfrak{sl}_2$ namely $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.
- Given parabolic $G \supset P \supset H$ consider $X = G/P$ with induced H action. For $[P] \in X^H$ the tangent space can be identified $\mathfrak{g}/\mathfrak{p}$ with adjoint action of H. If $\mathfrak{g}_\alpha \not\subset \mathfrak{p}$ then $\mathfrak{s}_\alpha$ descends to an H invariant $\mathbb{P}^1_\alpha \subset G/H$. These are eigencurves through $[P]$.
- Roots define H-compass at $[P]$. The reflection with respect to a root α defines transformation of the root system and, as an element of Weyl group $N(T)/T$ acting by conjugation on G/P which moves $[P]$ to $[P']$ which we will call a reflection as well.
- If the action of H on G/P is strong GKM then the shift along $\mathbb{P}^1_\alpha$ coincides with the differential of this morphism.
- This is not the case if the action is not strong GKM, e.g. $E_6(4)$.

$\mathbb{C}^*$ quotients

Suppose we have a subgroup $\lambda : \mathbb{C}^* \hookrightarrow H$ with quotient H' and $M' = \ker \lambda^* \subset M$. We consider quotients of X wrt the action of λ.

Note: if Y denotes any (geometric) GIT quotient and $\mathcal{Y}$ normalized Chow quotient then they admit induced H' action. Moreover if the action of H on X is k-GKM then the action of H' on Y and $\mathcal{Y}$ is $(k-1)$-GKM (but are usually singular).

Suppose that the action of H is GKM and C is an eigencurve which is the closure of an H orbit C° contained in the set of stable points of λ with fixed point $p \in C \cap X^H$.

Using local linearization of the action of H at p we identify a neighbourhood of p with $\mathbb{C}[x_1, x_2, \ldots x_n]$ with variables x_i graded in M. We may assume $C = \{x_2 = \cdots = x_n = 0\}$.

GIT quotient

In the situation from the previous slide, the GIT quotient around $[C^\circ]$ is (up to local linearization) described by the ring of invariants

$$\mathbb{C}[x_1^{\pm 1}, x_2, \ldots, x_n]^\lambda \subset \mathbb{C}[x_1^{\pm 1}, x_2, \ldots, x_n]$$

with the grading in M'.

If $\widetilde{M}$ denotes the lattice of characters of the big torus of $\mathbb{C}^n$ with basis $\deg x_i = v_i$ and grading map $\widetilde{M} \to M$ then the ring of invariants is

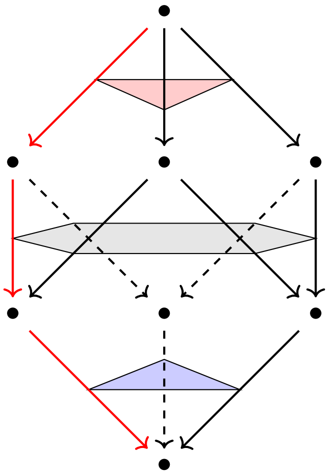
$$\mathbb{C}\left[\mathit{cone}(\pm v_1, v_2, \ldots, v_n) \cap \widehat{M}\right]$$

where $\widehat{M}$ is the kernel of composition $\widetilde{M} \longrightarrow M \xrightarrow{\lambda^*} \mathbb{Z}$.

In particular, if $\lambda^*(v_1) = \pm 1$ (i.e. the action λ has no nontrivial isotropy) then the ring is

$$\mathbb{C}\left[\left\langle v_i - \lambda^*(v_1)\lambda^*(v_i) \cdot v_1,\ i = 2, \ldots, n\right\rangle\right]$$

an example: $(\mathbb{P}^1)^{\times 3}$ with diagonal action



Arrows are eigencurves, horizontal slices are geometric GIT's wrt projection to vertical line, red path is a point in the Chow quotient.

the hypercube

(1) gens at 1st point of path $(e_1, e_2, e_3, \dots, e_n)$,
projection to section $(e_2 - e_1, e_3 - e_1, \dots, e_n - e_1)$,
invariants $\mathbb{C}[x_2x_1^{-1}, x_3x_1^{-1}, \cdots, x_nx_1^{-1}] \subset \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$

(2) gens at 2nd point of path $(-e_1, e_2, e_3, \dots, e_n)$,
projection to section $(e_2 - e_1, e_3 - e_2, \dots, e_n - e_2)$
invariants $\mathbb{C}[x_2x_1^{-1}, x_3x_2^{-1}, \cdots, x_nx_2^{-1}] \subset \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$

(3) gens at 2nd point of path $(-e_1, -e_2, e_3, \dots, e_n)$,
projection to section $(e_2 - e_1, e_3 - e_2, \dots, e_n - e_3)$
invariants $\mathbb{C}[x_2x_1^{-1}, x_3x_2^{-1}, \cdots, x_nx_3^{-1}] \subset \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$

$\cdots\cdots\cdots$

Note that the universal object containing all of the above is a monoid in M' generated by $e_i - e_j$ for $i > j$, or

$$\mathbb{C}[x_2x_1^{-1}, x_3x_2^{-1}, \cdots, x_nx_{n-1}^{-1}] \subset \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$$

normalized Chow quotient

Assume that the action is strong GKM. Suppose that $C = \bigcup_{i=1}^m C_i$ is a union of eigencurves, which is a closure of a general orbit of $\lambda : \mathbb{C}^* \hookrightarrow H$; each C_i is a closure of an orbit C_i° with source in p_{i-1} and sink in p_i.

- We can use shifts along C_i's to identify (linearized) local coordinate rings at $p_0, \ldots, p_{m-1}$ with subrings of Laurent polynomial ring $\mathbb{C}[\widetilde{M}]$; say $v_1^i, \ldots, v_n^i \in \widetilde{M}$ are generators of the coordinate ring at p_i.
- Recall $\widetilde{M} \longrightarrow M \xrightarrow{\lambda^*} \mathbb{Z}$ and $\widehat{M} = \ker \lambda^*$. If v_1^i represents the generator along C_{i+1} then the semi-group $\Lambda_i = cone(\pm v_1^i, \ldots, v_n^i) \cap \widehat{M} \subset \widetilde{M}$ yields GIT quotients étale local coordinate ring $\mathbb{C}[\Lambda_i]$ at the orbit C_i°.

normalized Chow quotient, cntd

- If $\Lambda = \sum_i \Lambda_i \subset \widehat{M}$ then $\mathbb{C}[\Lambda] \subset \mathbb{C}[\widehat{M}]$ is étale local coordinate ring of normalized Chow quotient $\mathcal{Y}$ at $[C]$.
- Morphisms of $\mathcal{Y}$ to a (geometric) GIT quotient containing the orbit C_i° are locally described by inclusion $\mathbb{C}[\Lambda_i] \subset \mathbb{C}[\Lambda]$.
- The induced grading $\Lambda \subset \widehat{M} \to M'$ determines the action of the quotients torus around $[C]$, hence the compass if $\mathcal{Y}$ is smooth at $[C]$.
- $\mathcal{Y}$ is usually not smooth at $[C]$; but if the action has no nontrivial isotropy and TX is nef (X is homogeneous) then it is.

line bundles on the normalized Chow quotient

Given a line bundle $\mathcal{O}(d)$ with $d \geq 1$ on $\mathbb{P}^1$ with the standard $\mathbb{C}^*$ action the space of sections splits into weight spaces

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(d)) = \bigoplus_{i=0}^{d} \mathbb{C}_{d_0+i}$$

with d_0 depending on the linearization and $\mu_{\mathcal{O}(d)}(\infty) = d_0$ and $\mu_{\mathcal{O}(d)}(0) = d_0 + d$.

Suppose that the degree of L on a general orbit of the action λ is d. Consider the incidence

$$X \xleftarrow{q} \mathcal{U} \xrightarrow{p} \mathcal{Y}$$

If P is flat then $p_* q^* L$ is locally free of rank $d+1$ and it splits into a sum of line bundles

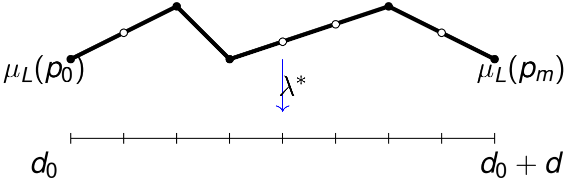
$$p_* q^* L = \bigoplus_{i=0}^{d} L_{d_0+i}$$

where d_0, $d_0 + d$ is the weight of the action of λ on a fiber of L over the sink and the source, respectively.

multigrid

Let C_i be the closure of a 1-dimensional orbit of the action of the torus H with stabilizer $H'' \subset H$. Then the weights of the action of the quotient torus is a rank 1 sub-lattice of M.
If the weights of the action of H on L over p_{i-1} and p_i are $u_{i-1}, u_i \in M$, respectively, then the action of H on $\mathrm{H}^0(C_i, L)_{|C_i}$ are $u_{i-1}, u_{i-1} + \delta_i, \ldots, u_i$ where $\delta_i = (u_i - u_{i-1})/\deg_{C_i} L$.

If $C = \bigcup_{i=1}^m C_i$ is the sum eigencurves, then weights of the action on L_C can be computed by calculating values over C_i's:



here you need to assume that components of the cycle are non-reduced, e.g. λ has non nontrivial isotropy

example of output, case of $C_4(4)^\dagger$

Chow quotient $\mathcal{Y}$ of 4th weight action on $C_4(4)$ is smooth of dimension 9 and it has 66 fixed points of the rank 3 torus action. For example, a point associated to a chain of three orbits, 2 lines and a conic, is described by weights of the action on fibers of line bundles

$$\begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix}$$

and the weights of the action on the cotangent space

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ -1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 1 & 0 & 2 \\ 1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

† this is the case of complete quadric