Birational geometry via $\mathbb{C}^*$ action

Beijing, April 2024

acknowledgments, earlier works

- This talk is in relation to joined projects with Mateusz Michałek, Leonid Monin, Gianluca Occhetta, Eleonora Romano, and Luis Solá Conde
- Variation of GIT for $\mathbb{C}^*$ action produce birational modifications: Thaddeus, Reid (1990's) and others
- Algebraic cobordism yields (weak) factorization of birational maps: Włodarczyk, Morelli (2000).
- Context: Mori Dream Spaces and Cox Rings; Hu, Keel, Cox, Hausen and others (2000+).

fundamental example: classical Cremona

Take classical Cremona transformation:

$$\mathbb{P}^2 \ni [z_0, z_1, z_2] \longrightarrow [z_1 z_2, z_0 z_2, z_0 z_1] = [z_0^{-1}, z_1^{-1}, z_2^{-1}] \in \mathbb{P}^2$$

Take product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with non-homogeneous coordinates (z_0, z_1, z_2) and $\mathbb{C}^*$ action, with $t \in \mathbb{C}^*$:

$$\boxed{t \cdot (z_0, z_1, z_2) \longrightarrow (tz_0, tz_1, tz_2)}$$

If $z_i \neq 0, \infty$ for $i = 0, 1, 2$ then

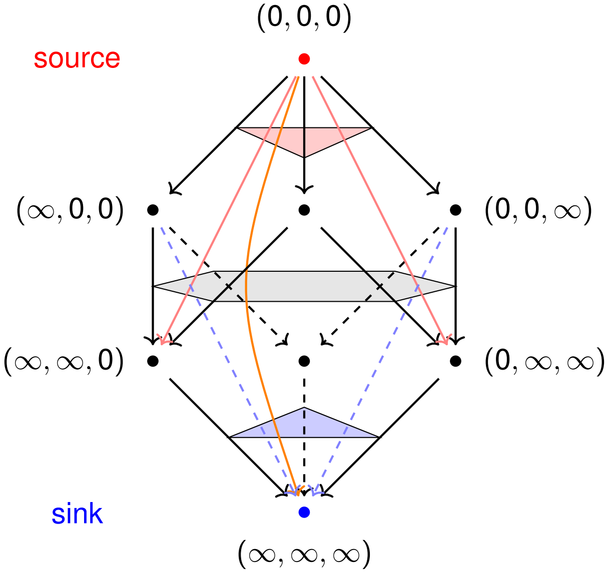
$$\lim_{t\to 0} t(z_0, z_1, z_2) = (0, 0, 0) \qquad \lim_{t\to\infty} t(z_0, z_1, z_2) = (\infty, \infty, \infty)$$

$$\frac{\partial t(z_0, z_1, z_2)}{\partial t}\Big|_{t=0} = (z_0, z_1, z_2) \qquad \frac{\partial t(z_0, z_1, z_2)}{\partial t}\Big|_{t=\infty} = (z_0^{-1}, z_1^{-1}, z_2^{-1})$$

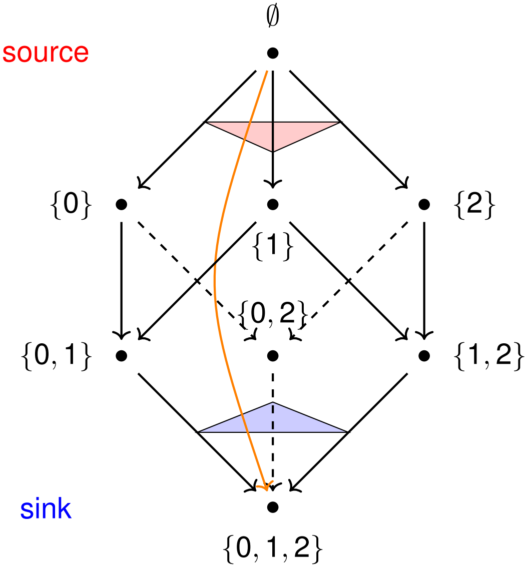
So we have a description of Cremona in terms of $\mathbb{C}^*$ action:

tangent to general orbit at 0 $\longrightarrow$ tangent to general orbit at ∞

orbits of action, sections

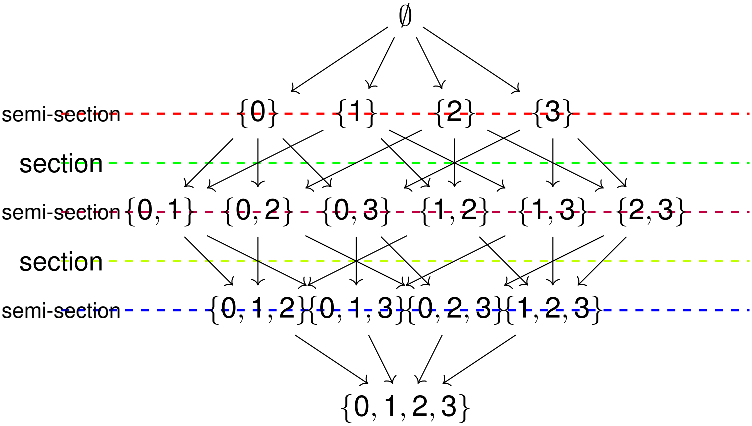


change of notation

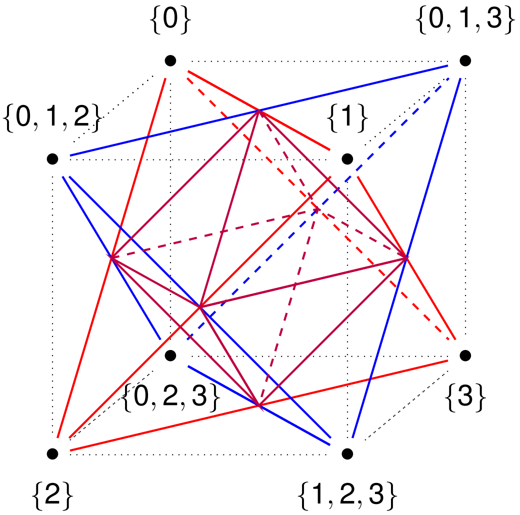


the same for higher Cremona

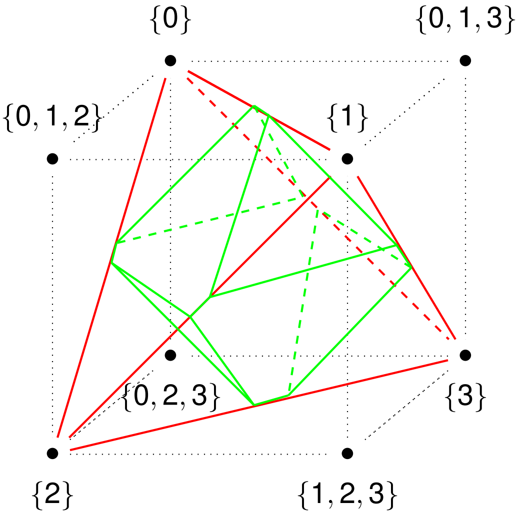
Take hypercube with vertices labeled by subsets of $\{0, \ldots, n\}$.
The diagonal action of $\mathbb{C}^*$ on $(\mathbb{P}^1)^{\times(n+1)}$ determines partial order of fixed points wich agrees with inclusion.
Section: division into two subsets which agrees with the order.



schematic picture of semi-sections

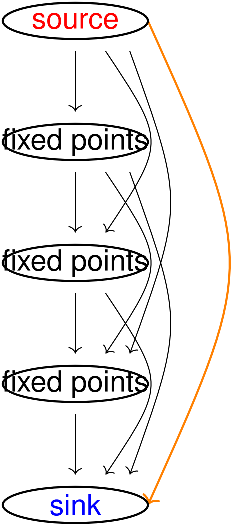


a section, truncated tetrahedron



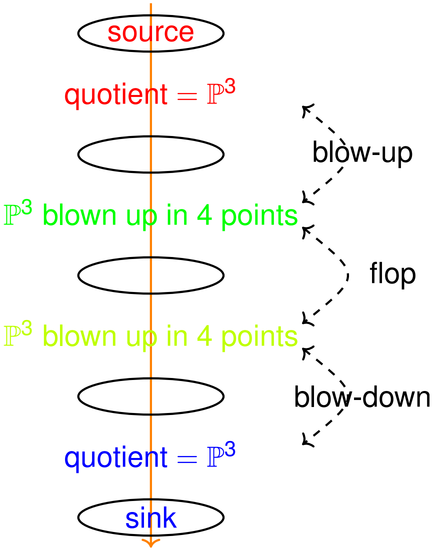
overview: GIT, choice of orbits

We choose an ample line bundle on $(\mathbb{P}^1)^{\times(n+1)}$ and change linarization of $\mathbb{C}^*$ action. This way we choose nontrivial orbits which we parametrize by the quotient. Remember that each nontrivial orbit starts and ends at the fixed point, generic orbit starts at source, ends in sink.



variation of Mumford's GIT, sections

ABB approach:
in case of $\mathbb{C}^*$ action quotients come from sections (geometric quotients) and semi-sections (good quotients) of the set of fixed points.



$\mathbb{C}^*$ action, linearization, fixed points

Set-up (X, L) smooth projective variety and an ample line bundle with $\mathbb{C}^*$ action $\mathbb{C}^* \times X \ni (t, x) \to t \cdot x \in X$ and linearization

$$\begin{array}{ccc} \mathbb{C}^* \times L & \xrightarrow{\mu} & L \\ \downarrow & & \downarrow \\ \mathbb{C}^* \times X & \longrightarrow & X \end{array}$$

The linearization yields decomposition into weight spaces

$$\mathrm{H}^0(X, L) = \bigoplus_{u \in \mathbb{Z}} \mathrm{H}^0(X, L)^u$$

$\mathrm{H}^0(X, L)^u$ the weight space associated to $u \in \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$.

For a fixed point component $F \subset X^{\mathbb{C}^*}$ define critical value

$$\mu(F) \in \mathrm{Aut}(L_y) = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z} \;\text{ where }\; y \in F$$

Number of $\mu(F)$'s will be called criticality of the action.

quotients: GIT and BB

GIT: fix $u \in \mathbb{Q}$ and consider

$$A_u = \bigoplus_{mu \in \mathbb{Z}} \mathrm{H}^0(X, mL)^{mu}$$

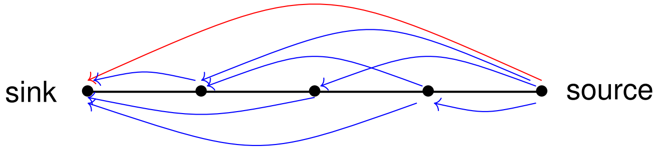
If u is well chosen we have rational quotient map

$$X \overset{\pi_u}{\dashrightarrow} \mathcal{Y}_u = \mathrm{Proj}(A_u)$$

BB: we have partial order on the set of fixed components

$$F_1 \prec F_2 \iff \exists x \in X : \lim_{t \to \infty} t \cdot x \in F_1, \ \lim_{t \to 0} t \cdot x \in F_2$$

The order agrees with the one induced by the linearization μ



unstable locus

To each component $F \subset X^{\mathbb{C}^*}$ we define associated BB cells

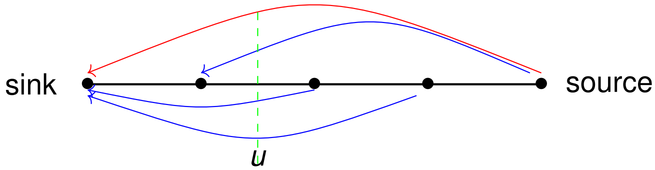
$$B^{\pm}(F) = \{x \in X : \lim_{t^{\pm 1} \to 0} t \cdot x \in F\}$$

Given $u \in \mathbb{Q}$ the quotient map

$$X \overset{\pi_u}{\dashrightarrow} \mathcal{Y}_u = \mathrm{Proj}(A_u)$$

is regular on complement of the set

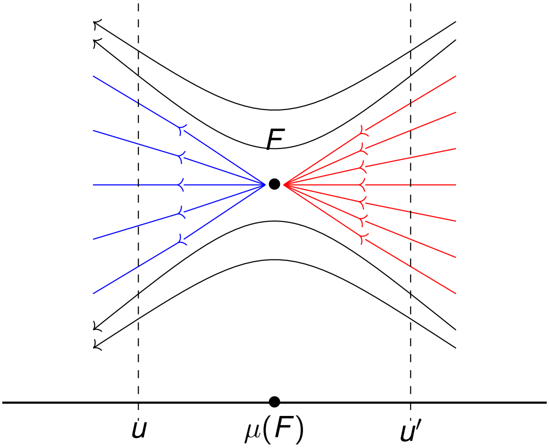
$$\left(\bigcup_{\mu(F) < u} B^{+}(F) \right) \cup \left(\bigcup_{\mu(F) > u} B^{-}(F) \right)$$



If u is not a critical value then the quotient is geometric.

variation of GIT

Changing linearization $u \mapsto u'$ determines the change of the geometry of the quotient



equalized bordism type action

Assumptions:

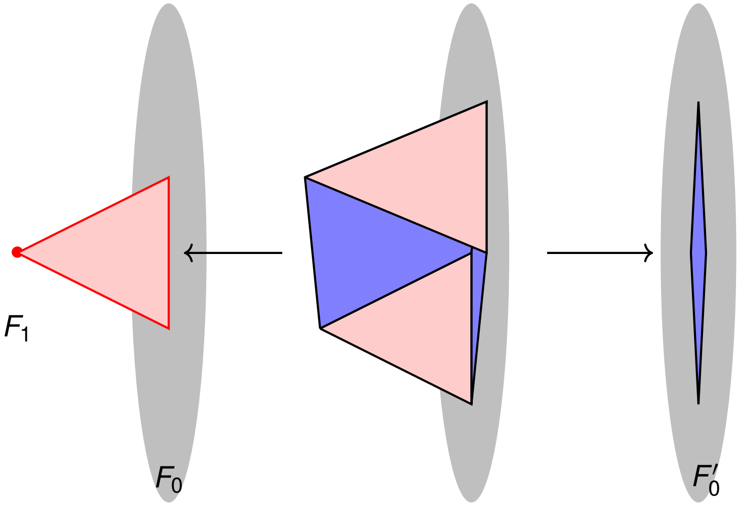
- The action is equalized: no point has non-trivial finite isotropy; equivalently ,weights of the action at fixed points are in $\{-1, 0, +1\}$
- The source and sink fixed point components are divisors F_0 and F_∞.
- There is no other divisorial BB cell.

Observation: if $u \in (\mu(F_\infty), \mu(F_0))$ is integral then we have equality of subspaces of $\mathrm{H}^0(X, L)$:

$$\mathrm{H}^0(X, L)^u = \mathrm{H}^0(X, L \otimes \mathcal{O}(-a_0 F_0 - a_\infty F_\infty))$$

where $a_0 = \mu(F_0) - u$ and $a_\infty = u - \mu(F_\infty)$.

Flipping extremal BB cells



Flipping the first/last but one BB cell flips its intersection with the source/sink and reduces the criticality of the action.

modifications of the $\mathbb{C}^*$ variety

Let us take $u_\infty, u_0 \in \mathbb{Q}$ such that $\mu(F_\infty) \leq u_\infty \leq u_0 \leq \mu(F_0)$ and set $a_\infty = u_\infty - \mu(F_\infty)$, $a_0 = \mu(F_0) - u_0$.
Then

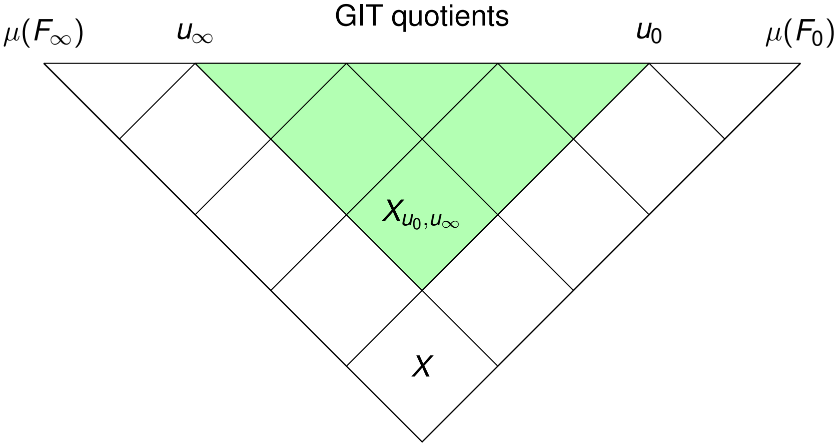
$$\mathrm{H}^0(X, L \otimes \mathcal{O}(-a_1 F_\infty - a_0 F_0)) = \bigoplus_{u \in [u_\infty, u_0]} \mathrm{H}^0(X, L)^u$$

Replacing $L \mapsto mL$ and u_0, u_∞ accordingly we get rational map

$$X \overset{\varphi_{u_0,u_\infty}}{\dashrightarrow} X_{u_0,u_\infty}$$

and the variety X_{u_0,u_∞} admits $\mathbb{C}^*$ action.

birational equivariant modification and GIT



Note that the top row birational modifications of X admit regular maps to their GIT quotients.

normalized Chow/Hilbert/universal quotients

Given the action $\mathbb{C}^* \times X \to X$ consider the maximal family of invariant 1-cycles containing a general orbit as a general point, by $\mathcal{C}$ we denote its normalization and $\mathcal{U}$ the universal family

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{p} & \mathcal{C} \\ \downarrow q & & \\ X & & \end{array}$$

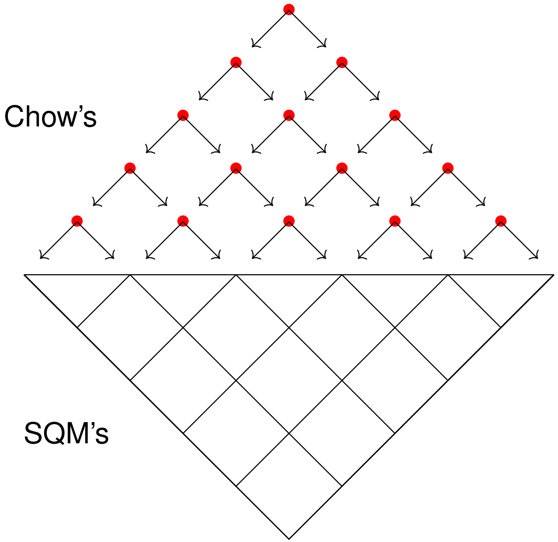
The morphism p is then flat and we have equivariant decomposition of a vector bundle over $\mathcal{C}$

$$p_* q^*(mL) = \bigoplus_{u \in \mathbb{Z}} \mathcal{L}_m^u$$

with $\mathrm{H}^0(\mathcal{C}, \mathcal{L}_m^u) = \mathrm{H}^0(X, mL)^u$ hence we have regular morphisms to GIT quotients

$$\mathcal{C} \longrightarrow \mathcal{Y}_u$$

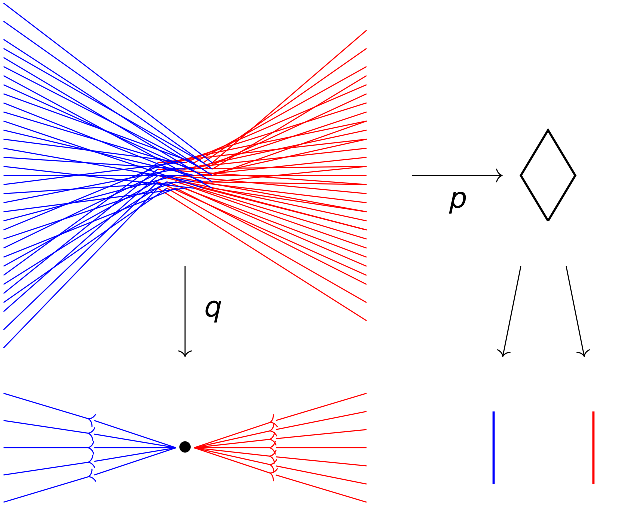
the system of normalized Chow quotients



Each • represents normalized Chow quotient for respective SQM of X. The bottom row are geometric GIT quotients.

resolving flips

The second from the bottom row of Chows resolves local flips arising from variation of GIT



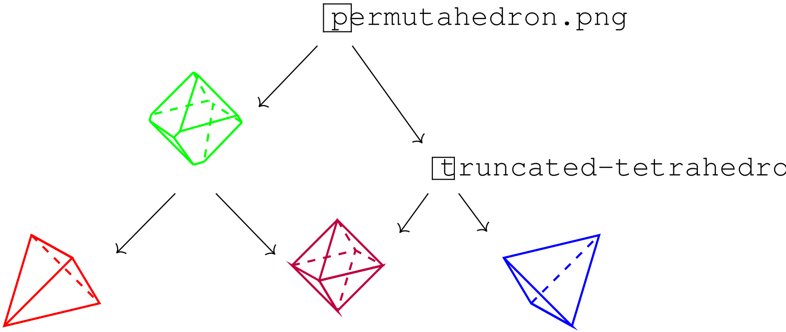
remarks, toric case

Note: (1) The construction of GIT quotients and equvariant SQM's above depend on the choice of L (up to its multiplicity) while normalized Chow does not. (2) If you allow divisorial BB cells (apart the source/sink) then you will have quotients which are not SQM's.

Toric example: diagonal $\mathbb{C}^*$ action on $(\mathbb{P}^1)^{\times n}$ with $L = \mathcal{O}(1, \ldots, 1)$ in toric terms represented by the projection of n-cube with projection orthogonal to the diagonal: the Chow quotient is a permutahedron the vertices of which are numbered by the sequences of edges in the cube.

permutahedron.png

case of toric varieties



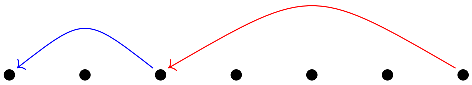
inversion of symmetric matrices

For $V \simeq \mathbb{C}^n$ we take the standard symplectic form on $V \oplus V^*$ and $\mathbb{C}^*$ action of weights ± 1 on V and V^*, respectively. The action lifts up to Lagrangian Grassmanian $X = LG(n, V \oplus V^*)$ with isolated source and sink, the blow-up to $F_0 \simeq F_\infty \simeq S^2(V)$ yields a rational map coming from inversion of symmetric matrices.

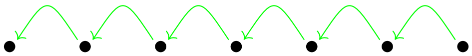
$\mathrm{Pic}\, X \simeq \mathbb{Z}$ is generated by Plücker line bundle L. The fixed point components are Grassmanian varieties of V with restriction of $L \simeq \mathcal{O}(2)$.

complete quadric

The Chow quotient of the action is obtained by blowing up in $\mathbb{P}(S^2V)$ locus of symmetric matrices of rank $1,\ 2, \ldots$ and a generic point of each exceptional divisor is associated to locus of union of two orbits stopping at a fixed point set component:



The divisors intersect transversally, their intersection parametrizes cycles consisting of smalles orbits



A similar situation occurs for equalized $\mathbb{C}^*$ action on homogeneous (or convex) varieties.