AIVAR notes

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1 Algorithm for symmetric tensor decomposition

1.1 The algorithm

In this section we will review the algorithm of symmetric tensor decomposition by B. Mourrain, J. Brachat, P.Comon and E. Tsigaridas first presented in [BCMT10]. We refer to the main paper as well as [BT20] for details on the proofs.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Following the notation in [BCMT10], we set $S = \mathbb{K}[x_0, ..., x_n]$ and $R = \mathbb{K}[x_1, ..., x_n]$. We will denote by S_d the space of homogeneous polynomials of degree $d \geq 0$ and $S_{\leq d}$ the space of polynomials of degree at most d, and similarly for R.

For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$, and $d \geq 0$, we will also use the multindex notation $\binom{d}{\alpha} := \frac{d}{\alpha!(d-|\alpha|)!}$, where $\alpha! = \alpha_1!...\alpha_n!$.

1.1.1 Reformulation in terms of the dual space

The first step is to reformulate the problem of Waring decomposition into the dual of the polynomial space.

Definition 1.1. *Let $F, G \in S_d$ with $F = \sum_{|\alpha|=d} f_\alpha x^\alpha$ and $G = \sum_{|\alpha|=d} g_\alpha x^\alpha$. We define the apolar inner product of F and G as*

$$\langle F, G \rangle := \sum_{|\alpha|=d} \frac{f_\alpha g_\alpha}{\binom{d}{\alpha}}$$

To every $F \in S_d$ we can associate an element in $(S_d)^* = Hom_{\mathbb{K}}(S_d, \mathbb{K})$ through the following map:

$$\tau : S_d \to (S_d)^*$$
$$\tau(F)(G) = \langle F, G \rangle$$

for every $G \in S_d$. We will denote $F^* = \tau(F)$.

Among the elements in S^* we find the evaluation maps: given $(\zeta_0, ..., \zeta_n) \in \mathbb{K}^{n+1}$, for every $F \in S$.

$$\mathbb{1}_{(\zeta_0,...,\zeta_n)}(F) = F(\zeta_0, ..., \zeta_n)$$

When there is no possible ambiguity we will abuse notation and use the same $\mathbb{1}_{(\zeta_0,...,\zeta_n)}$ for its restrictions to S_d or $S_{\le d}$.

Lemma 1.2. *Let $L = a_0x_0 + ... + a_nx_n \in S_1$. Then $\tau(L^d) = \mathbb{1}_{(a_0,...,a_n)} \in (S_d)^*$*

Proof. It is enough to check the image of a monomial. Let $x^\beta \in S_d$. Then

$$\tau(L^d)(x^\beta) = \langle L^d, x^\beta\rangle = \sum_{|\alpha|=d} \binom{d}{\alpha} \langle a_0^{\alpha_0}...a_n^{\alpha_n} x^\alpha, x^\beta\rangle = a_0^{\beta_0}...a_n^{\beta_n} = x^\beta(a_0, ..., a_n)$$

□

Now, since the assignment τ is injective, finding a Waring decomposition of a given polynomial $F \in S_d$, that is, finding $L_1, ..., L_r \in S_d$ such that $F = \sum_{i=1}^r L_i^d$, is equivalent to finding points $\zeta_1, ..., \zeta_r$ such that

$$F^* = \sum_{i=1}^r \mathbb{1}_{\zeta_i}$$

Remark 1.3. By a generic change of coordinates, we can assume that the linear forms L_i in the Waring decomposition of F are not in the hyperplane $x_0 = 0$, i.e. $L_i = a_{i,0}x_0 + ... + a_{i,n}x_n$ with $a_0 \ne 0$. In this case, we can write $F = \sum_i \lambda_i L_i^d$ for some $\lambda_i \in \mathbb{K}$ such that $a_{i,0} = 1$ for all i. Geometrically, this means that all the L_i lie in the same affine chart $x_0 \ne 0$.

If $F \in S_d$ and $g \in R_{\le d} = \mathbb{K}[x_1, ..., x_n]_{\le d}$, we denote by g^{h_d} the homogenization of g by the variable x_0 in degree d:

$$g^{h_d} := x_0^d g(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0})$$

Then we can define the form

$$f^* : R_{\le d} \to \mathbb{K}$$

$$g \mapsto \langle F, g^{h_d}\rangle$$

Expressing $F = \sum_i^r \lambda_i L_i^d$ with $L_i = x_0 + a_{i,1}x_1 + ... + a_{i,n}$ as noted in the remark above, using Lemma 1.2 we have

$$F^* = \sum_{i=1}^r \lambda_i \mathbb{1}_{(1,a_{i,1},...,a_{i,n})}$$

and therefore for every $g \in R_{\leq d}$

$$f^*(g) = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{(1,a_{i,1},\ldots,a_{i,n})}(g^h) = \sum_i \lambda_i g^h(1,a_{i,1},\ldots,a_{i,n}) = \sum_i \lambda_i g(a_{i,1},\ldots,a_{i,n})$$

Hence, under this localization

$$f^* = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{(a_{i,1},\ldots,a_{i,n})}$$

Note we have abused the notation for the evaluation forms, as in this case $\mathbb{1}_{(a_{i,1},\ldots,a_{i,n})} \in (R_{\leq d})^*$.

Thus, the problem of finding a Waring decomposition of $F \in S_d$ is equivalent (after a random change of variables if necessary) to finding $f \in (R_{\leq d})^*$ such that

$$f^* = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{(a_{i,1},\ldots,a_{i,n})}$$

1.1.2 Theoretical background

We start by stating a basic result on Artinian polynomial algebras:

Proposition 1.4 ([CLO07, § 5.3]). *Let $I \subseteq R$ be an ideal. The following are equivalent:*

i) I is a zero-dimensional ideal defining the affine variety $\{\zeta_1,\ldots,\zeta_r\}$.

ii) The algebra R/I is a vector space of finite dimension, say r'

Moreover, $r \leq r'$ and the equality holds if and only if I is radical.

Following the notation in [BCMT10], we make the space R^* into an R-module using the usual apolar action: for every $p \in R$, $\Lambda \in R^*$, we set

$$p \star \Lambda : R \to \mathbb{K}$$

$$q \mapsto \Lambda(p \cdot q)$$

One can easily check that expressing Λ as a power series in $K[[x_1,\ldots,x_n]]$ in the basis $\{\frac{x^\alpha}{\alpha!}\}_{\alpha\in\mathbb{N}^n}$, we have $p * \Lambda = p(\partial)(\Lambda)$ where $p(\partial)$ is obtained by substituting the variable x_i in p by the partial operator ∂_{x_i} (see [Polyexp] Prop 2.3).

In this language, the classical annihilator is given by the kernel of the following linear map:

Definition 1.5. *Fix $\Lambda \in R^*$. We define the Hankel operator associated to Λ H_Λ as*

$$H_\Lambda : R \to R^*$$

$$p \mapsto p \star \Lambda$$

Immediately from the definition we see

Lemma 1.6. *For every $\Lambda \in R^*$, the kernel of H_λ is an ideal of R.*

Proof. Let $p \in \ker(H_\Lambda)$ and $p \in R$. Then for every $h \in R$

$$H_\Lambda(qp)(h) = (qp) \star \Lambda(h) = \Lambda(pqh) = p \star \Lambda(qh) = 0$$

which shows $qp \in \ker(H_\Lambda)$. □

In order to be able to associate a matrix to the Hankel operator, we need to restrict it to a finite dimensional subspace. Given $B = \{b_1, ..., b_r\}, B' = \{b'_1, ..., b'_{r'}\} \subseteq R$, we define

$$H_\Lambda^{B,B'} : \langle B\rangle_{\mathbb{K}} \to \langle B'\rangle_{\mathbb{K}}^*$$

the restriction of H_Λ to $\langle B\rangle_{\mathbb{K}}$ and projection of R^* into $\langle B'\rangle_{\mathbb{K}}^*$. In this case, the matrix of $H_\Lambda^{B,B'}$ is

$$(H_\Lambda^{B,B'})_{i,j} = \Lambda(b_i \cdot b'_j) \qquad i = 1, ..., r,\ j = 1, ..., r'$$

If $B = B'$, we will use $H_\Lambda^{B,B'} = H_\Lambda^B$.

Remark 1.7. One can check that the catalecticant matrices correspond to the case where B, B' are the monomial of degree $\leq k$ and $\leq d - k$, respectively

We will state the three important results upon which the algorithm is based on. The first one directly relates the Hankel operator with the dual version of the Waring decomposition:

Theorem 1.8 ([BCMT10, Theorem 3.8])**.** *Let $\Lambda \in R^*$. Then $\Lambda = \sum_i^r \lambda_i \mathbb{1}_{\zeta_i}$ for $\lambda_i \neq 0$ and $\zeta_i \in \mathbb{K}^n$ distinct points if and only if rank $H_\Lambda = r$ and $\ker(H_\Lambda)$ is a radical ideal.*

Therefore we turn our attention to the algebra $A_\Lambda := R/\ker(H_\Lambda)$. The rank of H_Λ is the dimension of A_Λ as a $\mathbb{K}$-vector space.

For the second result, we need to introduce a key element in this theory, the multiplication operators:

Definition 1.9. *Let $a \in A_\Lambda$. We define the multiplication operator M_a by*

$$M_a : A_\Lambda \to A_\Lambda \qquad b \mapsto ab$$

$$M_a^t : A_\Lambda^* \to A_\Lambda^* \qquad \sigma \mapsto a \star \sigma$$

Our interest in these operators is highlighted by the following result:

Theorem 1.10 ([BCMT10, Theorem 3.10]). *Assume that A_Λ is a finite dimensional vector space. Then $\Lambda = \sum_{i=1}^r \mathbb{1}_{\zeta_i} \circ p_i(\partial)$ for some $\zeta_i \in \mathbb{K}^n$ and $p_i(\partial) \in \mathbb{K}[\partial_1, ..., \partial_n]$ and*

i) For every $a \in A_\Lambda$, the eigenvalues of the operators M_a and M_a^t are $\{a(\zeta_i), ..., a(\zeta_r)\}$.

ii) The common eigenvectors of $M_{x_i}^t$, $i = 1, ..., n$ are, up to scalar, $\mathbb{1}_{\zeta_i}$.

Therefore, on a computational level reading the points of the evaluation forms can be reduced to an eigenvector problem of the multiplication operators of the coordinate polynomials. In turn, these can be read directly from the Hankel, as the following identity shows:

Lemma 1.11. *For any $\Lambda \in R^*$ such that rank $H_\Lambda < \infty$ and any $a \in A_\Lambda$,*

$$H_{a\star\Lambda} = M_a^t \circ H_\Lambda$$

Proof. We begin by showing that this equality is well-defined, since we defined M_a^t in A_Λ^* and H_Λ in R^*. Let $p + h \in R$ be a representative of $a \in A_\Lambda$ with $h \in Ker(H_\Lambda)$. Then for every $q \in R$ $H_\Lambda(p+h)(q) = (p \star \Lambda)(q) + (h \star \Lambda)(q) = (p \star \Lambda)(q) = H_\Lambda(p)(q)$. Now we can check the equality above: for every $p \in R$,

$$H_{a\star\Lambda}(p) = p \star a \star \Lambda = a \star p \star \Lambda = M_a^t(H_\Lambda(p))$$

□

When working with truncated Hankel operators, if we want to use this formula we need H_Λ to be invertible to compute the matrix M_a^t. This condition is equivalent to finding a basis of A_Λ:

Proposition 1.12 ([BT20, Proposition 3.2]). *Let $B \subseteq A_\Lambda$ with $|B| = r$. Then B is a basis of the vector space A_Λ if and only if H_Λ^B is invertible and rank $H_\Lambda = r$.*

Thus, given a form $\Lambda \in R^*$ finding the points corresponding to the evaluation forms of its decomposition amounts to finding a basis of the algebra H_Λ, which will allow us to compute the multiplication operators and its eigenvectors. By Proposition 1.4, the radicality of $\ker(H_\Lambda)$ simply consists in checking that the number of common eigenvectors of $M_{x_i}^t$ is the same as the dimension of A_Λ.

1.1.3 Extending f^*

A crucial observation is that the algebraic results we have described so far require an element in $\Lambda \in R^*$. However, given a poynomial $F \in S_d$ for which we want to find the Waring decomposition, we have only associated a form $f^* \in (R_{\leq d})^*$. The core idea of the algorithm is to extend this form into R^*.

More precisely, given $f^* \in (R_{\leq d})^*$ we want to find $\Lambda \in R^*$ such that $\Lambda = \sum_{i=1}^r \lambda_i \mathbb{1}_{(a_{i,1},\ldots,a_{i,n})}$ and $\Lambda|_{R_{\leq d}} = f^*$. This decomposition of Λ automatically gives the desired Waring decomposition of f^* by restricting the evaluation maps to $(R_{\leq d})^*$.

Fixing the monomial basis $\{x^\alpha\}_{\alpha\in\mathbb{N}^n}$ of R^* and the dual basis of R^* $\{\frac{1}{\alpha}x^\alpha\}_\alpha$, for every $\Lambda \in R^*$ the matrix of the Hankel operator H_Λ is $(\Lambda(x^{\alpha+\beta}))_{\alpha,\beta\in\mathbb{N}^n}$. Now if we want Λ to extend f^* we impose that it agrees with $f^* \in (R_{\leq d})^*$ up to degree d, and leave the rest of the coefficients of the matrix as unknowns. Hence, we write the matrix of H_Λ as

$$(H_\Lambda)_{\alpha,\beta} = \begin{cases} f^*(x^{\alpha+\beta}) & \text{if } |\alpha+\beta| \leq d \\ h_{\alpha+\beta} & \text{if } |\alpha+\beta| > d \end{cases} \tag{1}$$

The third important result describes the conditions of the variables $h's$ under which we have the existence of such $\Lambda \in R^*$ extending f^*, and guarantees uniqueness. We need a previous definition:

Definition 1.13. *Let $B \subseteq R$ be a set of monomials. We say B is connected to 1 if $\forall m \in B$ either $m = 1$ or there is $1 \leq i \leq n$ and $m' \in B$ such that $m = x_i m'$.*

For example, $B = 1, x_1, ..., x_n, x_1^2, x_1x_3$ is connected to 1. For any $B \subseteq R$, we denote $B^+ := B \cup x_1 B \cup ... \cup x_n B$.

Theorem 1.14 ([BCMT10, Theorem 4.2])**.** *Let $B = \{x^{\beta_1}, ..., x^{\beta_r}\}$ be a set of monomials of degree at most d connected to 1 and $\Lambda \in \langle B \cdot B^+\rangle^*_{\leq d}$. Let $\Lambda(h)$ be a form in $\langle B \cdot B^+\rangle^*$ defined by $\Lambda(h)(x^\alpha) = \Lambda(x^\alpha)$ if $|\alpha| \leq d$ and h_α otherwise. Then $\Lambda(h)$ admits an extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B a basis of $A_{\tilde{\Lambda}}$ if and only if*

$$M^B_{x_i}(h) \circ M^B_{x_j}(h) - M^B_{x_j}(h) \circ M^B_{x_i}(h) = 0$$

for all $i, j = 1, ..., n$ and the determinant of $H^B_\Lambda(h)$ is non-zero. Moreover, $\tilde{\Lambda}$ is unique.

Roughly speaking, this theorem states that, once we have a basis of our algebra, the choice of the variables h's in Eq. 1 is obtained by imposing the commutation of the multiplication operators of the variables.

Remark 1.15. Let e denote the maximum degree of the monomials in B of the theorem. If $2e + 1 \leq d$, then $\Lambda(h) = \Lambda$. The multiplication operators will only contain numerical values, and therefore if these operators commute then the extension of the operator $\tilde{\Lambda}$ is unique and there are no choices of h's involved.

The last theoretical remark is which is the minimal possible rank, that is, which is the first value of the rank that has to be tested in the algorithm. We define $H_f^{\circledast}$ as the largest numerical submatrix of $H_{\Lambda(h)}$, which is easily checked to be of size $\binom{n+\lceil d/2\rceil}{n} \times \binom{n+\lfloor d/2\rfloor}{n}$. If the Waring rank of F is r, then $r \geq \operatorname{rank} H_f^{\circledast}$. Indeed, $H_f^{\circledast}$ is a submatrix of $H_{\Lambda(h)}$, which is of rank r by Theorem 1.8, so $H_f^{\circledast}$ cannot have bigger rank than r.

1.1.4 Description of the algorithm

We now have all the necessary ingredients to understand the steps of the algorithm:

Algorithm for Waring decomposition

Input: $F \in S_d$

Output: Waring decomposition of F.

1. Construct the matrix $H_{\Lambda(h)}$ with parameters $\{h_\alpha\}_{\alpha\in\mathbb{N}^n}$, $|\alpha| > d$.
2. Set $r = \operatorname{rank} H_f^{\circledast}$.
3. For $B \subseteq R$ a set of monomials connected to 1 and $|B| = r$, do:
   - Find h's such that:
     - $H_{\Lambda(h)}^B$ has nonzero determinant
     - The multiplication operators $(M_{x_i})^t$ commute for all $i = 1, ..., n$.
     - There are r eigenvectors $v_1, ..., v_r$ common to $M_{x_i}^t$, $i = 1, ..., n$.
   - If found, go to Step 5.
4. Set $r \to r + 1$ and go to Step 3.
5. Solve the linear system $F = \sum_i^r \lambda_i (x_0 + v_{i,1} + ... + v_{i,n} x_n)^d$ to find λ_i.

Even though in some sense this algorithm and the results behind solve theoretically the problem of finding a Waring decomposition of a homogeneous polynomial, the practical implementation of the algorithm is far from being accomplished. In the cases where no choice of the variables h's is needed, the complexity of the algorithm is reduced to linear algebra computations, and there are currently published codes in different mathematical softwares that run the algorithm in this case. However, when choices of h's have to be made the condition on the commutativity of the matrix multiplication operators give rise to polynomial equations that are usually impractical to solve.

It is interesting to note that every choice of h's verifying the required conditions gives rise in general to a different Waring decomposition, and therefore with this algorithm we obtain all the Waring decompositions except those that include linear forms in the hyperplane $x_0 = 0$.

In [BT20] we find a significant improvement on the possible choices of the basis B. It is proven that there is a monomial basis with elements of degree at most d of A_Λ that is a *complete staircase*, which is a more restrictive condition than connected to 1:

Definition 1.16. *Let $B \subseteq R$ be a set of monomials. We say that B is a staircase if for all $i = 1, ..., n$ $x_i x^\beta \in B$ implies $x^\beta \in B$. If in addition B contains all the degree one monomials we say that B is a complete staircase.*

Let us see an an application with a concrete example.

1.1.5 Example

Consider the polynomial $F = -4xy + 2xz + 2yz + z^2 \in S_3$. The associated form in $R_{\leq 3}$ is $f^* = -4y + 2z + 2yz + z^2$. Let $\Lambda \in R^*$ the form extending f, we know some of its entries:

	1	y	z	y^2	yz	z^2
1	$f^*(1)$	$f^*(y)$	$f^*(z)$	$f^*(y^2)$	$f^*(yz)$	$f^*(z^2)$
y	$f^*(y)$	$f^*(y^2)$	$f^*(yz)$			
z	$f^*(z)$	$f^*(yz)$	$f^*(z^2)$			
y^2	$f^*(y^2)$					
yz	$f^*(yz)$					
z^2	$f^*(z^2)$					

After some calculations we obtain

	1	y	z	y^2	yz	z^2
1	0	-2	1	0	1	1
y	-2	0	1	0		
z	1	1	1			
y^2	0					
yz	1					
z^2	1					

The rest of the values are filled with variables (taking into account the symmetries):

	1	y	z	y^2	yz	z^2
1	0	-2	1	0	1	1
y	-2	0	1	$h_{3,0}$	$h_{2,1}$	$h_{1,2}$
z	1	1	1	$h_{2,1}$	$h_{1,2}$	$h_{0,3}$
y^2	0	$h_{3,0}$	$h_{2,1}$	$h_{4,0}$	$h_{3,1}$	$h_{2,2}$
yz	1	$h_{2,1}$	$h_{1,2}$	$h_{3,1}$	$h_{2,2}$	$h_{1,3}$
z^2	1	$h_{1,2}$	$h_{0,3}$	$h_{2,2}$	$h_{1,3}$	$h_{0,4}$

So far this is the first step in the algorithm described before. The first value of r to iterate is 3, since the principal $3\times$ 3 minor is nonzero. Our guess for B is $B = \{1, y, z\}$, which is indeed a complete staircase and

$$H_\Lambda^B = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Next we build the multiplication by y and z operators using Lemma 1.6. Directly from the computed Hankel we see

$$H^B_{y*\Lambda} = \begin{pmatrix} -2 & 0 & 1 \\ 0 & h_{3,0} & h_{2,1} \\ 1 & h_{2,1} & h_{1,2} \end{pmatrix}.$$

and so we have

$$M^B_y = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3}{8}h_{3,0} + \frac{1}{4}h_{2,1} & \frac{1}{8}h_{3,0} + \frac{1}{4}h_{2,1} & \frac{1}{4}h_{3,0} + \frac{1}{2}h_{2,1} \\ -\frac{3}{8}h_{2,1} + \frac{1}{4}h_{1,2} + \frac{1}{8} & \frac{1}{8}h_{2,1} + \frac{1}{4}h_{1,2} - \frac{3}{8} & \frac{1}{4}h_{2,1} + \frac{1}{2}h_{1,2} + \frac{1}{4} \end{pmatrix}$$

Similarly,

$$M^B_z = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{3}{8}h_{2,1} + \frac{1}{4}h_{1,2} + \frac{1}{8} & \frac{1}{8}h_{2,1} + \frac{1}{4}h_{1,2} - \frac{3}{8} & \frac{1}{4}h_{2,1} + \frac{1}{2}h_{1,2} + \frac{1}{4} \\ -\frac{3}{8}h_{1,2} + \frac{1}{4}h_{0,3} + \frac{1}{8} & \frac{1}{8}h_{1,2} + \frac{1}{8}h_{0,3} - \frac{3}{8} & \frac{1}{4}h_{1,2} + \frac{1}{2}h_{0,3} + \frac{1}{4} \end{pmatrix}$$

Imposing that the multiplication operators commute, which in this case is the single relation $M^B_z M^B_y = M^B_y M^B_z$, yields the equation

$$h_{3,0}h_{1,2} + 2h_{3,0}h_{0,3} - 3h_{3,0} - h^2_{2,1}$$
$$-2h_{1,2}h_{2,1} + 4h_{2,1}h_{0,3} + 8h_{2,1} - 4h^2_{1,2} - 4h_{1,2} - 1 = 0$$

Which has solution

$$h_{3,0} = -2 \quad h_{0,3} = 4 \quad h_{2,1} = 1 \quad h_{1,2} = 1$$

From here we can compute the eigenvectors of M^t_y and M^t_z:

$$(M^B_y)^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with eigenspaces} \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$(M^B_z)^t = \begin{pmatrix} 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} \\ 1 & 1 & \frac{5}{2} \end{pmatrix} \quad \text{with eigenspaces} \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$$

Common eigenspaces:

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$$

Solve in λ_i:

$$f^* = -4y + 2z + 2yz + z^2 = \lambda_1 (1 - y + 0z)^2 + \lambda_2 \left(1 + y - \frac{1}{2}z\right)^2 + \lambda_3 (1 + y + 3z)^2.$$

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{8}{7}, \quad \lambda_3 = 17.$$

Hence, we have

$$f^* = -4y + 2z + 2yz + z^2 = (1 - y + 0z)^2 + \frac{-8}{7}\left(1 + y - \frac{1}{2}z\right)^2 + 17\,(1 + y + 3z)^2$$

We can recover our initial polynomial by homogenizing:

$$f = -4xy + 2xz + 2yz + z^2 = (x - y + 0z)^2 + \frac{-8}{7}\left(x + y - \frac{1}{2}z\right)^2 + 17\,(x + y + 3z)^2$$

1.2 Reinterpreting the algorithm

1.2.1 Translating the terms

Keeping the same notation, we can summarize the previous discussion in the following "sequence" of transformations:

$$S_d \to (S_d)^* \to (R^*_{\leq d)} \to R^* \to S^*$$
$$F \mapsto F^* \mapsto f^* \mapsto \Lambda \mapsto \tilde{F}$$

Let us clarify the last steps: given $F \in S_d$, the algorithm produces a form $\Lambda = \sum_i^r \lambda_i \mathbb{1}_{\zeta_i} \in R^*$ with $\zeta_i = (\zeta_{i,1}, ..., \zeta_{i,n}) \in \mathbb{K}^n$, such that its restriction to $(R_{\leq d})$ is equal to f^*. According to the localization when defining f^*, the way to globalize the functional $\Lambda \in R^*$ is simply to embed the points ζ_i in the affine chart $x_0 \neq 0$ of the projective space. That is,

$$\tilde{F} = \sum_i^r \lambda_i \mathbb{1}_{(1,\zeta_{i,1}...,\zeta_{i,n})}$$

Note that by Lemma 1.2 when restricting the evaluation functions above to S_d we recover F^*.

Our goal is to extend F^* directly to $\tilde{F}$ without the intermediate steps, which involve defining the apolar inner product in Definition 1.1.

Let $S = Sym(V^*) = \mathbb{K}[\alpha_0, ..., \alpha_n]$. Then there is an isomorphism

$$\mathbb{K}[[\alpha_0, ..., \alpha_n]] \to Hom_{\mathbb{K}}(S, \mathbb{K}) \qquad \sum_I a_i \alpha^{(I)} \mapsto (\alpha^{(I)} \mapsto a_i)$$

Note that if $l = \zeta_1\alpha_1 + ... + \zeta_n\alpha_n$ then evaluation maps $\mathbb{1}_{(\zeta_1,...,\zeta_n)} \in S^*$ correspond to the power series

$$\exp l := 1 + l + l^{(2)} + l^{(3)} + ...$$

Indeed, for every $\phi = \sum_I b_I \alpha^I \in S$

$$\exp(l)(\phi) = \sum_I b_i \exp(l)(\alpha^I) = \sum_I b_I l^{(|I|)}(\alpha^{(I)}) = \phi(\zeta_0, ..., \zeta_n)$$

So the previous algorithm takes $F \in S_d$ and produces an element $\tilde{F} = \sum_i \lambda \exp(l_i) \in S^*$. By truncating the power series we obtain $\tilde{F} = \tilde{F}_0 + \tilde{F}_1 + ... + \tilde{F}_d + ... \in S^*$ with $\tilde{F}_i \in S_i^*$ and $\tilde{F}_d = F^*$.

In this setting, if $\hat{T}$ denotes the power series ring, the apolar action can be formulated as

$$S \otimes \hat{T} \to \hat{T}$$
$$\phi \times F \mapsto (\psi \mapsto F(\phi \cdot \psi))$$

In the notation of the previous section, the image of $\phi \otimes F$ is $H_F(\phi)$, where H_F denotes the Hankel operator.

1.2.2 The algorithm as an apolar integral

We start with two claims we proved. Fix $F = \sum_i^r \lambda_i l_i^d$ and let $\lrcorner$ denote the apolar action. We define the usual annihilator

$$Ann(\tilde{F}) \coloneqq \{\phi \in S \mid \phi \lrcorner \tilde{F} = 0\}$$

Lemma 1.17.

$$Ann(\tilde{F}) = I(\{l_1, ..., l_r\})$$

Proof. Let $\phi \in S$ such that $\phi(l_i) = 0$ for all i. Then for every $g \in S$ we have

$$(\phi \lrcorner \tilde{F})(g) = \sum_i^r \lambda_i \exp(l_i)(\phi \cdot g) = \sum_i^r \lambda_i \exp(l_i)(\phi) \cdot \exp(l_i)(g) = 0$$

For the other inclusion, let $\phi \in Ann(\tilde{F})$ and take p_i an interpolation polynomial of the points $l_1, ..., l_r$, at l_i, that is, $p_i(\zeta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then

$$0 = \phi \lrcorner \tilde{F}(p_i) = \sum_i^r \lambda_i \exp(l_i)(\phi \cdot p_i) = \phi(l_i)$$

which proves the other inclusion. □

Lemma 1.18. *For all j, $Ann(\tilde{F}) \cap S^d(V^*) \subseteq Ann(\tilde{F}_j)$*

Proof. If $\phi \in Ann(\tilde{F}) \cap S^d(V^*)$ then expanding $\tilde{F}$ into its graded components and applying linearity automatically $\phi \lrcorner \tilde{F}_j = 0$ for all j. □

We observe that $\alpha_0 \lrcorner \tilde{F}_{d+1} = \tilde{F}$. Indeed, if $l = \alpha_0 + \zeta_1\alpha_1 + ...\zeta_n\alpha_n$ as in the algorithm, for every $g \in S$

$$\alpha_0 \lrcorner \exp(l)(g) = \exp(l)(\alpha_0)\exp(l)(g) = \exp(l)(g)$$

We note here the importance of $\exp(l_i)(\alpha_0) \neq 0$ for all i. Thus, in general we would look for $\alpha \in S_1$ such that $\alpha \lrcorner \tilde{F}_{d+1} = \tilde{F}_d = F^*$. We can express this as an "apolar integral"

$$\tilde{F}_{d+1} = \int F^* d\alpha + \text{ term in } \alpha^\perp$$

We could obtain the next piece $\tilde{F}_{d+2}$ integrating the resulting $\tilde{F}_{d+1}$.

We can end this process at $\tilde{F}_{2r}$, for if $2rk(F) \leq d$ then the Waring decomposition of $F = \sum_i l_i^{(d)}$ is unique and "easy" to find:

$$\mathcal{I}([l_1], ..., [l_r]) = (Ann(F)_{\leq r})$$

1.2.3 A worked example

We can test this method with a well-known example: the normal form a smooth cubic

$$F = x(x-z)(x+z) - y^2 z$$

which we know it has rank 4.

In divided powers, this expression divided by two becomes

$$F^* = \tilde{F}_3 = 3x^{(3)} - xz^{(2)} - y^{(2)}z$$

Integrating with respect to x yields

$$\tilde{F}_4 = 3x^{(4)} - x^{(2)}z^{(2)} - xy^{(2)}z + h_0 y^{(4)} + h_1 y^{(3)}z + \dots + h_4 z^{(4)}$$

Since we want $\tilde{F}_4$ to have the same exponentials, we impose that it has rank 4. We know that the Waring rank is greater or equal than the rank of the biggest square catalecticant matrix $Cat^{\lfloor d/2 \rfloor, \lceil d/2 \rceil}$. (see e.g. [T14]).

	$x^{(2)}$	xy	xz	$y^{(2)}$	yz	$z^{(2)}$
$x^{(2)}$	3	0	0	0	0	−1
xy	0	0	0	0	−1	0
xz	0	0	−1	−1	0	0
$y^{(2)}$	0	0	−1	h_0	h_1	h_2
yz	0	−1	0	h_1	h_2	h_3
$z^{(2)}$	−1	0	0	h_2	h_3	h_4

Imposing the rank to be 4, we find

$$-3h_0 + 3 = 0 \quad h_2 = 0 \quad -3h_4 + 1 = 0$$

with h_1, h_3 free parameters, i.e., an affine variety of dimension 2.

Thus, we can say that if $F = \sum_i^4 \lambda_i l_i^{(4)}$ with l_i not in $\alpha^\perp$, then there exists h_1, h_3 such that

$$\tilde{F}_4 = 3x^{(4)} - x^{(2)}z^{(2)} - xy^{(2)}z - y^{(4)} + h_1 y^{(3)}z + h_3 yz^{(3)} + \frac{1}{3}z^{(4)}$$

Let us go through the next integral: we look for $\tilde{F}_5$ such that $\alpha_0 \lrcorner \tilde{F}_4 = \tilde{F}_5$. Hence,

$$\tilde{F}_5 = 3x^{(5)} - x^{(3)}z^{(2)} - x^{(2)}y^{(2)}z - xy^{(4)} + h_1 xy^{(2)}z + h_3 xyz^{(3)} + \frac{1}{3}xz^{(4)} + h_5 y^{(5)} + \dots + h_{10} z^{(5)}$$

We compute the largest square catalecticant, $Cat^{\lfloor d/2 \rfloor, \lceil d/2 \rceil} = Cat^{3,2}$:

	$x^{(3)}$	$x^{(2)}y$	$x^{(2)}z$	$xy^{(2)}$	xyz	$xz^{(2)}$	$y^{(3)}$	$y^{(2)}z$	$yz^{(2)}$	$z^{(3)}$
$x^{(2)}$	3	0	0	0	0	−1	0	−1	0	0
xy	0	0	0	0	−1	0	−1	h_1	0	h_3
xz	0	0	−1	−1	0	0	h_1	0	h_3	$\frac{1}{3}$
$y^{(2)}$	0	0	−1	0	h_1	0	h_5	h_6	h_7	h_8
yz	0	−1	0	h_1	0	h_3	h_6	h_7	h_8	h_9
$z^{(2)}$	−1	0	0	0	h_3	$\frac{1}{3}$	h_7	h_8	h_9	h_{10}

It is important to note that the principal 6×6 submatrix agrees with the previous catalecticant. This is a mere consequence of the fact that we have multiplied the polynomial $\tilde{F}_4$ by x and this submatrix corresponds to the monomial basis of degree 2 multiplied by x.

Imposing the rank to be ≤ 4, we find a much larger polynomial system:

$$\begin{cases} -3h_3 + 3h_7 = 0 \\ 3h_1h_3 - 3h_3h_5 + 3h_1h_7 = 0 \\ -2h_1 + h_5 = 0 \\ h_9 = 0 \\ 9h_8^2 - 9h_6h_{10} - 6h_8 + 1 = 0 \\ 6h_7h_8 - 3h_5h_{10} - 2h_7 = 0 \\ h_7^2 + h_{10} = 0 \\ 6h_6h_7 - 3h_5h_8 + h_5 = 0 \\ 3h_5h_7 + 6h_8 - 2 = 0 \\ h_5^2 + 4h_6 = 0 \end{cases}$$

After examining this ideal with a mathematical software for symbolic computation, we determined that it is prime and it still has dimension 2

We have also implemented the algorithm of the previous section for this polynomial. After dehomogenizing we are left with two variables, for which there are only three possible complete staircase basis of 4 elements:

$$\{\{1, y, z, y^2\}, \{1, y, z, yz\}, \{1, y, z, z^2\}\}.$$

Only $(1, y, z, yz)$ gives us a non-zero determinant on the corresponding minor of the Hankel. For this case, the commutation of the multiplication operators (which in this case is the single matrix equation $M_yM_z = M_zM_y$) is associated with the following ideal (using the same variables as before):

$$I = (h_2, h_3 - h_7, 27h_7h_1 + 9h_8 - 1, 27h_3h_1 + 9h_8 - 1)$$

One can check that this ideal, up to divided powers, is equal to the one obtained with the new approach.

1.2.4 Next steps

We can formalize the "apolar integral" that allows us to obtain $\tilde{F}_{d+1}$ from $\tilde{F}_d$ above with the following map:

$$S^{d-1}V \otimes V^* \to S^d V$$

$$(G, \alpha) \mapsto \alpha \lrcorner G$$

Since the kernel is non-trivial, it induces the rational map

$$\mathbb{P}(S^d V) \dashleftarrow \mathbb{P}(S^{d-1}V) \times \mathbb{P}(V^*)$$

If we restrict it to the linear subset of sums of r points $\sigma_r^o(d)$ (i.e., the r-th secant variety "without the closure"), we have

$$\sigma_r^o(d) \dashleftarrow \sigma_r^o(d+1) \times \mathbb{P}(V^*)$$

Note that the map is surjetive, since we can take the Waring decomposition of F and raise the powers of the linear forms by one. We can take similar restrictions to the secant variety $\sigma_r(d)$ and the cactus $k_r^o(d)$:

$$\sigma_r(d) \dashleftarrow \sigma_r(d) \times \mathbb{P}(V^*)$$

$$k_r^o(d) \dashleftarrow k_r^o(d) \times \mathbb{P}(V^*)$$

One would like to understand better these maps.

Lastly, an open question that was discussed is the following: if $I \subseteq Sym(V)$ is an ideal, then $\alpha \lrcorner (I_{d+1})^\perp \subseteq (I_d)^\perp$. But if I is saturated, α is general, is it true that $\alpha \lrcorner (I_{d+1})^\perp = (I_d)^\perp$?

2 Schur apolarity

In this section, we will review the *Schur apolarity theory*, first presented by R. Staffolani in [S18]. We refer to the main paper for a more in-depth discussion.

Let V be a vector space of dimension $n+1$ over an algebraically closed field K of characteristic zero. Recall that the irreducible representations of $\mathrm{SL}(V)$ are given by the Schur modules $\mathbb{S}_\lambda V$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of the integer $d = |\lambda|$ and $k \leq n+1$. Our goal is to extend the classical apolarity approach – used, for instance, to study Waring decompositions of homogeneous polynomials – to the setting of an arbitrary Schur module $\mathbb{S}_\lambda V$.

In this context a decomposition of a tensor $f \in \mathbb{S}_\lambda V$ as a sum of "rank–one" elements (i.e. points of the minimal $SL(V)$–orbit, which are – as we shall see – flag varieties) is equivalent to an inclusion of certain "apolar" ideals. The Schur apolarity action will be our main tool.

2.1 Preliminaries

We begin by setting the notation and recalling the construction of Schur modules.

2.1.1 Schur Modules via Young Symmetrizers

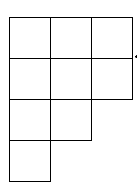
Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of the integer $d = |\lambda|$, meaning that

$$d = \lambda_1 + \lambda_2 + \dots + \lambda_k \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0.$$

The *Young diagram of* λ is a collection of boxes arranged in left-justified rows, where the i-th row contains λ_i boxes. The integer k is called the *length of the partition* λ, and it is denoted by $l(\lambda)$.

A *Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers. It is called *standard*, if it is filled with the numbers $1, 2, \dots, d$, with entries strictly increasing from left to right in each row and from top to bottom in each column; *semi–standard*, if the entries are weakly increasing from left to right in each row and strictly increasing from top to bottom in each column.

Example 2.1. . Let $\lambda = (3, 3, 2, 1)$. Its Young diagram is



.

A standard tableau of shape λ is

$$T = \begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 & 9 \\ \hline 3 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

Given a standard tableau T of shape λ, denote by R_T the subgroup of the symmetric group $\mathfrak{S}_{|\lambda|}$ preserving the rows and by C_T the subgroup preserving the columns. The *Young symmetrizer associated to* λ *and* T is the endomorphism c_λ^T of $V^{\otimes d}$ that sends the tensor $v_1 \otimes \cdots \otimes v_d$ to

$$\sum_{\tau \in C_\lambda} \sum_{\sigma \in R_\lambda} \operatorname{sign}(\tau) v_{\tau(\sigma(1))} \otimes \cdots \otimes v_{\tau(\sigma(d))}.$$

The image of the Young symmetrizer

$$\mathbb{S}_\lambda^T V \coloneqq c_\lambda^T (V^{\otimes d})$$

is the *Schur module* associated to λ. It is not difficult to prove that any two Schur modules that are images of two different standard tableaux of shape λ, are isomorphic. Thus, we drop the T and we write $\mathbb{S}_\lambda V$. It can be seen that Schur modules are irreducible representations of $SL(V)$ through the induced action of the group. They are completely determined by the partition λ, cf. ([FH13], p. 77). From the definition of the Young symmetrizer, we have

$$\mathbb{S}_\lambda V \subset \bigwedge^{\lambda_1'} V \otimes \cdots \otimes \bigwedge^{\lambda_h'} V =: \bigwedge_{\lambda'} V,$$

where λ' is the *conjugate* partition to λ, obtained by simply transposing the diagram of λ.

Example 2.2. Let $\lambda = (3, 2, 1)$. A standard tableau of shape λ is

$$T = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

The image $c_{(3,2,1)}^T (V^{\otimes 6})$ is the module $\mathbb{S}_{(3,2,1)} V$. A generic element of $\mathbb{S}_{(3,2,1)} V$ can be written as a linear combination of "simple" tensors of the form $(v_1 \wedge v_2 \wedge v_3) \otimes (v_4 \wedge v_5) \otimes v_6$.

Remark 2.3. Although any two standard tableaux of shape λ yield isomorphic Schur modules, their embeddings in $V^{\otimes |\lambda|}$ are generally different.

Remark 2.4. For a positive integer d, consider the partitions $\lambda = (d)$ and $\mu = (1^d) \coloneqq (1, \ldots, 1)$. Then we have $\mathbb{S}_\lambda V = \operatorname{Sym}^d V$ and $\mathbb{S}_\mu V = \bigwedge^d V$.

2.1.2 Tensors parametrized by flag varieties

We now want to provide a geometric interpretation of the "simplest" tensors that arise from Schur modules.

Let T be a standard tableau of shape λ, with $|\lambda| = d$, and let $\{v_1, \dots, v_n\}$ be a basis for the vector space V. A pair (T, S) – where S is a semi–standard tableau of the same shape as T – is called a *bitableau*. Bitableaux represent elements of $\mathrm{S}_\lambda V$ as follows. First, define the tensor

$$\mathbf{v}_{(T,S)} \coloneqq v_{i_1} \otimes \cdots \otimes v_{i_d} \in V^{\otimes d},$$

where the index assignment is such that $v_{i_j} = v_k$ if there is a k in the box of S corresponding to the box of T in which a j appears. When the choice of T is unambiguous, we may simply denote the tensor by $\mathbf{v}_S$. Next, by applying the Young symmetrizer c_λ^T to $\mathbf{v}_S$, we obtain an element of $\mathrm{S}_\lambda V$.

Example 2.5. Take $\lambda = (2, 2, 1)$, and consider the standard and semi–standard tableaux

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad S = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then, $\mathbf{v}_S = v_1 \otimes v_2 \otimes v_3 \otimes v_1 \otimes v_2$ and $c^T(\mathbf{v}_S) = (v_1 \wedge v_2 \wedge v_3) \otimes (v_1 \wedge v_2)$. Note that a different choice of the standard tableau T gives the same element $c^T(\mathbf{v}_S) \subset \bigwedge^3 V \otimes \bigwedge^2 V$, although the latter embeds differently in $V^{\otimes 5}$.

Remark 2.6. Since we are interested in studying the elements of $\mathbb{S}_\lambda V$ independently from how they embed in $V^{\otimes d}$, we reduce to work over the vector space

$$\mathbb{S}^\bullet V \coloneqq \bigoplus_\lambda \mathbb{S}_\lambda V.$$

We simplify the theory by constructing each Schur module using a fixed standard tableau. For a partition λ of the integer d, this tableau is created by filling the diagram of λ from top to bottom, beginning with the first column, using the integers $1, \dots, d$.

In order to understand how flag varieties are related to tensors, we need the following result.

Lemma 2.7. *Let V be an n-dimensional vector space and let λ be a partition whose length is strictly less than n. Then, up to scalar multiples, the highest weight vector of the representation $\mathrm{S}_\lambda V$ is obtained by applying the Young symmetrizer to the element corresponding to the semi–standard tableau $L(\lambda)$ of shape λ, where the i-th row is entirely filled with i's.*

For a detailed proof, see ([F97], p. 113). Consider the orbit $\mathrm{SL}(V)\cdot\mathbf{v}$, where $\mathbf{v}$ is the highest weight vector of $\mathrm{S}_\lambda V$. It is possible to show that this orbit is isomorphic to the quotient $\mathrm{SL}(V)/P$, where P is a parabolic subgroup of $\mathrm{SL}(V)$. It follows that $\mathrm{SL}(V)\cdot\mathbf{v}$ is a rational homogeneous variety, i.e. a flag variety.

Explicitly, if $\lambda = \left(\lambda_1^{a_1},\dots,\lambda_k^{a_k}\right)$ with $\lambda_1 > \cdots > \lambda_k > 0$ and $a_1+\cdots+a_k < \dim(V)$, then the flag variety embedded in $\mathbb{P}(\mathrm{S}_\lambda V)$ is

$$X = \mathbb{F}(n_1,\dots,n_k;V) \;\coloneqq\; \{(V_1,\dots,V_k) : V_1 \subset \cdots \subset V_k \subset V,\ \dim(V_i) = n_i\},$$

with the embedding defined by the line bundle $\mathcal{O}(d_1,\dots,d_k)$. Here, the integers n_i and d_i satisfy

$$n_i = \sum_{j=1}^{i} a_j \quad \text{and} \quad d_i = \lambda_i - \lambda_{i+1},$$

with the convention $\lambda_{k+1} = 0$, for $i = 1,\dots,k$. Note that the conjugate partition to λ is precisely $\lambda' = \left(n_k^{d_k},\dots,n_1^{d_1}\right)$. The points of X-rank 1 are of the form

$$(v_1 \wedge \cdots \wedge v_{n_k})^{\otimes d_k} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1},$$

representing the flag

$$\langle v_1,\dots,v_{n_1}\rangle \subset \cdots \subset \langle v_1,\dots,v_{n_k}\rangle.$$

In this context, the X-rank is called λ-*rank*.

Example 2.8. Let V be a complex vector space of dimension 7, with basis $\{v_1,...,v_7\}$. Consider the partition $\lambda = (6,4^3,1^2)$ and a standard tableau T of shape λ. We have

$$L(\lambda) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 & 4 & 4 \\ \cline{1-4} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}.$$

Hence, the highest weight vector of $S_\lambda V$ is

$$\mathbf{t} = c_\lambda^T(\mathbf{v}_{L(\lambda)}) = (v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge v_5 \wedge v_6) \otimes (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{\otimes 3} \otimes v_1.$$

Considering its orbit via the action of $\mathrm{SL}(V)$, we obtain $X = \mathbb{F}(1,4,6)$, embedded in $\mathbb{P}(\mathbb{S}_\lambda V)$ with $\mathcal{O}(1,3,1)$. Therefore, a point $p \in X$ will be of the form

$$< v_1 > \subset < v_1, v_2, v_3, v_4 > \subset < v_1, v_2, v_3, v_4, v_5, v_6 > .$$

2.2 Symmetric and Skew–symmetric apolarities

In the following section, we briefly recall the main concepts and notations of the symmetric apolarity theory, also known as classical apolarity, and the skew-symetric apolarity theory.

2.2.1 Symmetric apolarity

Let V be a vector space of dimension $n+1$ over an algebraically closed field K of characteristic zero. Consider the symmetric algebra $S := \mathrm{Sym}(V)$ and its dual algebra $R := \mathrm{Sym}(V^*)$. Recall that we can consider S as the ring of homogeneous polynomials $S = K[x_0, ..., x_n]$, with dual ring $R = K[y_0, ..., y_n]$ acting on S by differentiation:

$$y_j \cdot x_i = \frac{\partial}{\partial x_j}(x_i) = \delta_{ij}.$$

For any homogeneous polynomial $f \in S$, its *apolar ideal* is defined by

$$f^\perp = \{g \in R \mid g(f) = 0\}.$$

This ideal is a fundamental tool in the study of the Waring problem for polynomials: one seeks the minimal expression $f = \sum_{i=1}^r a_i\, l_i^d$, with linear forms $l_i \in V$ and scalars $a_i \in K$. In geometric terms, if we consider the projectivization, the set of points

$$X = \{[l_1^*], \ldots, [l_r^*]\} \subset \mathbb{P}(V^*)$$

has the property that its homogeneous ideal $I_X \subset R$ is contained in $f^\perp$. This fact is encapsulated in the classical:

Lemma 2.9 (Apolarity Lemma, cf. [IK99], [D04])**.** *A homogeneous polynomial $f \in S$ of degree d can be written as $f = \sum_{i=1}^r a_i\, l_i^d$ if and only if the homogeneous ideal $I_X \subset R$ of the scheme $X = \{[l_1^*], \ldots, [l_r^*]\} \subset \mathbb{P}(V^*)$ satisfies $I_X \subset f^\perp$.*

In practice, one uses *catalecticant matrices* associated with the polynomial f. For each $0 \le i \le d$, define $C_f^{i,d-i}: S^i V^* \to S^{d-i} V$, by

$$C_f^{i,d-i}(y_0^{i_0} \cdots y_n^{i_n}) = \frac{\partial^i f}{\partial x_0^{i_0} \cdots \partial x_n^{i_n}},$$

where $\sum_{j=0}^n i_j = i$. The ranks of these matrices yield important information on the decomposition of f.

2.2.2 Skew–Symmetric apolarity

Now we turn our attention to the skew–symmetric setting, first introduced in [ABMM21]. Let $\bigwedge^d V$ denote the d–th exterior power of V. A tensor of the form $\mathbf{v} = v_1 \wedge v_2 \wedge \cdots \wedge v_d \in \bigwedge^d V$, is said to be of *skew–symmetric rank one*. The Grassmannian $\mathbb{G}(d, V) \subset \mathbb{P}(\bigwedge^d \mathbb{V})$ is precisely the set of projectivized rank–one tensors.

To mimic the apolarity action in this setting we define a skew–symmetric pairing. Let $\mathbf{h} = h_1 \wedge h_2 \wedge \cdots \wedge h_i \in \bigwedge^i V^*$ and $\mathbf{v} = v_1 \wedge v_2 \wedge \cdots \wedge v_i \in \bigwedge^i V$. Their pairing is defined by

$$\mathbf{h}(\mathbf{v}) = \det\Big[h_j(v_k)\Big]_{1\leq j,k\leq i}.$$

For any $\mathbf{v} = v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$ and $\mathbf{h} = h_1 \wedge \cdots \wedge h_s \in \bigwedge^s V^*$, with $0 \leq s \leq d$, we define

$$\mathbf{h} \cdot \mathbf{t} = \sum_{\substack{R\subset\{1,\ldots,d\}\\ |R|=s}} \operatorname{sign}(R)\, \mathbf{h}(\mathbf{v}_R)\, \mathbf{v}_{\overline{R}},$$

where $\overline{R} = \{1, \ldots, d\} \smallsetminus R$, $\operatorname{sign}(R)$ is the sign of the permutation that sends $\{1, ..., d\}$ in $R \cup \overline{R}$, $\mathbf{v}_R$ denotes the wedge product of the $\mathbf{v}_i$ for indices $i \in R$ and $\mathbf{v}_{\overline{R}}$ is the wedge product of the remaining factors. Extending this action by linearity, we define the *skew–apolarity action*.

Now, we can define the *skew–catalecticant map*, associated to any $\mathbf{t} \in \bigwedge^d V$:

$$\mathcal{C}_{\mathbf{t}}^{s,d-s} \colon \bigwedge^{s} V^* \longrightarrow \bigwedge^{d-s} V, \quad \mathcal{C}_{\mathbf{t}}^{s,d-s}(\mathbf{h}) = \mathbf{h} \cdot \mathbf{t}.$$

Then, we set

$$\mathbf{t}^\perp = \bigoplus_{s=0}^{d} \ker\big(\mathcal{C}_t^{s,d-s}\big),$$

which is interpreted as the skew–symmetric analogue of the apolar ideal. Moreover, for any skew-symmetric rank–one tensors $\mathbf{w}_i \in \bigwedge^d V$, we define the associated skew–symmetric ideal

$$I^\wedge(\mathbf{w}_1, \ldots, \mathbf{w}_r) \coloneqq \bigcap_{i=1}^{r} (\mathbf{w_i})^\perp,$$

where $(\mathbf{w}_i)^\perp$ denotes the subspace orthogonal to the one represented by $[\mathbf{w}_i] \in \mathbb{G}(d, V)$.

With the above constructions, we recover the analogue of the Apolarity–Lemma for the skew–symmetric case.

Lemma 2.10 (Skew-Apolarity Lemma, cf. ([ABMM21], Lemma 12))**.** *Let* $\mathbf{v}_i = v_i^{(1)} \wedge \cdots \wedge v_i^{(d)} \in \mathbb{G}(d, V) \subset \bigwedge^d V$, *for* $i = 1, \ldots, r$, *and let* $\mathbf{t} \in \bigwedge^d V$. *Then the following statements are equivalent:*

1. *The tensor* $\mathbf{t}$ *can be decomposed as* $\mathbf{t} = \sum_{i=1}^{r} a_i\, \mathbf{v}_i$, *with scalars* $a_1, \ldots, a_r \in K$*;*
2. $I^\wedge(\mathbf{v}_1, \ldots, \mathbf{v}_r) \subset (\mathbf{t}^{\wedge\perp})$;
3. $I^\wedge(\mathbf{v}_1, \ldots, \mathbf{v}_r)_d \subset (\mathbf{t}^{\wedge\perp})_d$.

2.3 Schur Apolarity Theory

In this section we develop the apolarity framework for general Schur modules. We will thoroughly follow the treatment in [S18].

2.3.1 Skew diagrams and multiplication maps

We first need to recall some basic combinatorial facts.

Definition 2.11. *Let λ and ν be two partitions such that $|\nu| = d$, $|\lambda| = e$ and $0 \leq e \leq d$. We write $\lambda \subset \nu$ if $\lambda_i = \nu_i$ for each i, eventually setting some λ_i's equal to 0. In this case, it is well defined the skew-partition $\nu/\lambda = (\nu_i - \lambda_i)$. We can attach to ν/λ a Young diagram, by simply canceling out in the diagram of ν the boxes of the diagram of λ, all justified to the left. This will be called the skew-Young diagram of ν/λ.*

Example 2.12. Let $\nu = (4, 3, 1)$ and $\lambda = (3)$. The skew diagram ν/λ (together with a semi–standard skew tableau of shape ν/λ) is

$$\begin{array}{|c|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare & 1 \\ \hline 1 & 2 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array}\ .$$

Definition 2.13. *Fix a standard skew tableau of shape ν/λ. One can define, analogously to the usual Young symmetrizer, the operator*

$$c_{\nu/\lambda} : V^{\otimes(|\nu|-|\lambda|)} \longrightarrow V^{\otimes(|\nu|-|\lambda|)}.$$

The skew Schur module $\mathbb{S}_{\nu/\lambda}V$ *is then defined as the image of $c_{\nu/\lambda}$.*

To develop the theory, we need to introduce the multiplication maps

$$\mathcal{M}_{\nu}^{\lambda,\mu} : \mathbb{S}_{\lambda}V^* \otimes \mathbb{S}_{\mu}V^* \longrightarrow \mathbb{S}_{\nu}V^*.$$

Because $\mathrm{SL}(n)$ is reductive, every representation is completely reducible. Furthermore, the Littlewood–Richardson rule governs the decomposition of the tensor product of any two irreducible representations. In particular, for any two Schur modules we have

$$\mathbb{S}_{\lambda}V^* \otimes \mathbb{S}_{\mu}V^* \simeq \bigoplus_{\nu} N_{\nu}^{\lambda,\mu}\, \mathbb{S}_{\nu}V^*,$$

where the coefficients $N_{\nu}^{\lambda,\mu} = N_{\nu}^{\mu,\lambda} \geq 0$ are the Littlewood–Richardson coefficients. Specifically, if $N_{\nu}^{\lambda,\mu} \neq 0$, then the projection

$$\mathcal{M}_{\nu}^{\lambda,\mu} : \mathbb{S}_{\lambda}V^* \otimes \mathbb{S}_{\mu}V^* \longrightarrow \mathbb{S}_{\nu}V^*$$

is nontrivial, and the dimension of its space of equivariant maps is $N_\nu^{\lambda,\mu}$.

Schur modules arising from different standard skew tableaux of the same shape are isomorphic. Moreover, although these modules remain representations of $\mathrm{SL}(V)$, they are not necessarily irreducible. In fact, one finds that

$$\mathbb{S}_{\nu/\lambda}V \cong \bigoplus_{\mu} N_\nu^{\lambda,\mu}\,\mathbb{S}_\mu V.$$

See ([FH13], p.83) for more details. In our setting, we assume that the modules are constructed from a standard skew tableau of shape ν/λ, where the boxes are filled with the integers $1, 2, \ldots, |\nu| - |\lambda|$ in sequence, starting from the top of the first column.

Definition 2.14. *For a given skew tableau T of shape ν/λ, we define the* word associated to T *as the string obtained by reading the entries of T from left to right, starting with the bottom row. This word, denoted $w_1w_2\cdots w_k$, is called a* Yamanouchi word *(or* Y-word*, or* reverse lattice word*) if for each $s = 0, \ldots, k-1$, the subword $w_kw_{k-1}\cdots w_{k-s}$ contains the integer $i+1$ at most as many times as it contains the integer i. The* content *of T is the sequence $\mu = (\mu_1, \ldots, \mu_n)$, where μ_i is the number of times the entry i appears in T.*

Definition 2.15. *We call a skew tableau T of shape ν/λ and content μ a* Littlewood–Richardson skew tableau *if its associated word is a Yamanouchi word.*

Example 2.16. Let $\nu = (3, 3, 2, 1)$ and $\lambda = (2)$. Consider the following skew tableaux of shape ν/λ:

$$T_1 = \begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & 1 \\ \hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & 1 \\ \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}.$$

The words associated to T_1 and T_2 are respectively 3231221 and 3231121.
Only the second one is a Y-word and therefore, T_2 is a Littlewood-–Richardson skew tableau.

Proposition 2.17 (cf. ([F97], p. 64))**.** *Let λ, μ, and ν be partitions with $\lambda, \mu \subset \nu$ and $|\mu| + |\lambda| = |\nu|$. Then the number of Littlewood–Richardson skew tableaux of shape ν/λ with content μ is exactly $N_\nu^{\lambda,\mu}$.*

Remark 2.18. Recall that there is a one-to-one correspondence between standard Young tableaux of shape λ (filled with $1,\dots,|\lambda|$) and Yamanouchi words of length $|\lambda|$ with content λ, i.e. with λ_i times the integer i. In fact, one can define two mutually inverse maps

$$\alpha : \{\text{standard tableaux of shape } \lambda\} \longrightarrow \{\text{Y-words of length } |\lambda| \text{ and content } \lambda\},$$

$$\beta : \{\text{Y-words of length } |\lambda| \text{ and content } \lambda\} \longrightarrow \{\text{standard tableaux of shape } \lambda\}.$$

For a given standard tableau T, if a_l denotes the row in which the entry l appears, then $\alpha(T)$ is defined as the reversed sequence $(a_{|\lambda|},\dots,a_1)$. Conversely, suppose that $\underline{a}$ is a Yamanouchi word of length $|\lambda|$ with content λ. To define $\beta(\underline{a})$ we start by reversing $\underline{a}$, obtaining the sequence $(a_1,...,a_{|\lambda|})$. For any integer $i \in \{1,...,k\}$, we consider the subsequence

$$(a_1,...,a_{|\lambda|})_i := (a_{k_1},...,a_{k_{\lambda_i}})$$

given by all the i's with respect to the order with which they appear in $(a_1,...,a_{|\lambda|})$. We finally set the (i,j)-th entry of $\beta(\underline{a})$ equal to the index k_j appearing in the subsequence $(a_1,...,a_{|\lambda|})_i$.
For further details on these correspondences, we refer to [ABW82].

Example 2.19. Consider the standard tableau of shape $\lambda = (3,2,2)$:

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}.$$

By applying α we obtain the Y-word $\alpha(T) = 3213121$. Now, we apply β and show that we obtain T back. First, we consider the reverse word

$$\underline{a} = (a_1,a_2,a_3,a_4,a_5,a_6,a_7) = (1213121)$$

. The three subsequences are:

- $\underline{a}_1 = (a_1,a_3,a_5) = (1,1,1)$;
- $\underline{a}_2 = (a_2,a_6) = (2,2)$;
- $\underline{a}_3 = (a_4,a_7) = (3,3)$.

The indexes appearing in each string are precisely the filling of T, hence $\beta(\alpha(T)) = T$.

Definition 2.20 (cf. ([ABW82], p.257))**.** *Let T be a Littlewood–Richardson skew tableau of shape ν/λ and content μ. We define a new tableau T' (of the same shape, with a new content μ') as follows. Let $\underline{a}$ be the word associated to T. Then define the word*

$$\underline{a}' \coloneqq \alpha\big((\beta(\underline{a}))'\big),$$

where $\beta(\underline{a})'$ denotes the conjugate (i.e., the transpose) of the tableau $\beta(\underline{a})$. Finally, define T' as the skew tableau of shape ν/λ whose associated word is $\underline{a}'$.

Example 2.21. (cf. ([S18], p. 44)). Let $\nu = (3,2)$ and $\lambda = (1)$. Consider the skew tableau

$$T = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}.$$

In this example, the content is $\mu = (3,1)$ and the associated word is $\underline{a} = (1,2,1,1)$, which is a Yamanouchi word. We compute

$$\beta(\underline{a}) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad \beta(\underline{a})' = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}.$$

Applying α to $\beta(\underline{a})'$ gives $\underline{a}' = (3,1,2,1)$, and thus

$$T' = \begin{array}{|c|c|c|} \hline & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} \end{array}.$$

Definition 2.22 (cf. [ABW82], p. 258)**.** *Let λ, μ, and ν be partitions with $N_\nu^{\lambda,\mu} \neq 0$. Given the skew Young diagram ν/λ and a Littlewood–Richardson skew tableau T of shape ν/λ with content μ, we define a map*

$$\sigma_T : \nu/\lambda \longrightarrow \mu$$

by assigning to each box at position (i,j) in ν/λ the pair

$$\sigma_T(i,j) \coloneqq \big(T(i,j),\, T'(i,j)\big).$$

Remark 2.23. Recall that the Young symmetrizer $c_\lambda^T : V^{\otimes d} \to V^{\otimes d}$ can be decomposed as $c_\lambda^T = b_\lambda \cdot a_\lambda$, where a_λ symmetrizes along rows and b_λ skew-symmetrizes along columns. In particular,

$$a_\lambda(V^{\otimes d}) = \mathrm{Sym}_\lambda V \coloneqq \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V,$$

and

$$b_\lambda(V^{\otimes d}) = \wedge_{\lambda'} V,$$

yielding the inclusion $\mathbb{S}_\lambda V \subset \wedge_{\lambda'} V$ discussed earlier.

We are now ready to define the maps

$$\mathcal{M}^{\lambda,\mu}_{\nu,T} : \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V^*$$

that will be used in our Schur apolarity framework.

Definition 2.24 (cf. ([S18], p. 45)). *Let λ, μ, and ν be partitions with $N^{\lambda,\mu}_\nu \neq 0$, and fix a Littlewood–Richardson tableau T of shape ν/λ and content μ. Define the map*

$$\mathcal{M}^{\lambda,\mu}_{\nu,T} : \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V^*$$

as the composition of the following maps:

1. *First, identify $\mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^*$ with a subspace of $\operatorname{Sym}_\lambda V^* \otimes \operatorname{Sym}_\mu V^*$ by using the transpose of the symmetrization map*

$$a_\lambda \otimes a_\mu : \operatorname{Sym}_\lambda V^* \otimes \operatorname{Sym}_\mu V^* \longrightarrow \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^*.$$

2. *Next, apply the embedding on the second factor:*

$$\operatorname{Sym}_\mu V^* \longrightarrow (V^*)^{\otimes|\mu|} \longrightarrow (V^*)^{\otimes(\nu/\lambda)} \longrightarrow \operatorname{Sym}_{\nu/\lambda} V^*,$$

where the intermediate step rearranges the factors according to the correspondence determined by σ_T.

3. *Then, combine the factors using the natural multiplication maps*

$$\operatorname{Sym}^{\lambda_i} V^* \otimes \operatorname{Sym}^{\nu_i-\lambda_i} V^* \longrightarrow \operatorname{Sym}^{\nu_i} V^*,$$

obtaining an element of $\operatorname{Sym}_\nu V^*$.

4. *Finally, apply the skew-symmetrizing operator b_ν (the column-symmetrizing part of the Young symmetrizer) to project onto $\mathbb{S}_\nu V^*$.*

Example 2.25. (cf. ([S18], p. 45-46)). Consider the partitions $\lambda = (2,1)$, $\mu = (1,1)$, and $\nu = (3,2)$. In this situation $N^{(2,1),(1,1)}_{(3,2)} = 1$, so there is a unique Littlewood–Richardson skew tableau of shape $(3,2)/(2,1)$ with content $(1,1)$, namely

$$T = \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 \\ \cline{1-2}\end{array}.$$

An element

$$t = (v_1 \wedge v_2 \otimes v_3 - v_2 \wedge v_3 \otimes v_1) \otimes (\alpha \wedge \beta)$$

in $\mathbb{S}_{(2,1)}V \otimes \mathbb{S}_{(1,1)}V$ is first sent to

$$(v_1 v_3 \otimes v_2 - v_2 v_3 \otimes v_1) \otimes (\alpha \otimes \beta - \beta \otimes \alpha)$$

in $\mathrm{Sym}_{(2,1)} V \otimes \mathrm{Sym}_{(1,1)} V$. Next, the rearrangement dictated by σ_T identifies the second factor with $\mathrm{Sym}_{(3,2)/(2,1)} V$. (In this case, $\mathrm{Sym}_{(1,1)} V \cong V \otimes V$.) After applying the multiplication maps

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^1 V \longrightarrow \mathrm{Sym}^3 V, \qquad \mathrm{Sym}^1 V \otimes \mathrm{Sym}^1 V \longrightarrow \mathrm{Sym}^2 V,$$

the element becomes

$$v_1 v_3 \alpha \otimes v_2 \beta - v_1 v_3 \beta \otimes v_2 \alpha - v_2 v_3 \alpha \otimes v_1 \beta + v_2 v_3 \beta \otimes v_1 \alpha$$

in $\mathrm{Sym}_{(3,2)} V$. Finally, applying the skew-symmetrizing operator yields an element of $\mathbb{S}_{(3,2)} V$.

Remark 2.26. The map $\mathcal{M}^{\lambda,\mu}_{\nu,T}$ is $\mathrm{SL}(V)$–equivariant and, by construction, its image lies in $\mathbb{S}_\nu V$. Moreover, different choices of the Littlewood–Richardson skew tableau T produce linearly independent maps.

Remark 2.27. The space

$$\mathbb{S}^\bullet V \coloneqq \bigoplus_\lambda \mathbb{S}_\lambda V,$$

together with the multiplication maps $\mathcal{M}^{\lambda,\mu}_\nu$, forms a graded ring. More precisely, if we define the graded pieces by

$$(\mathbb{S}^\bullet V)_a \coloneqq \bigoplus_{\lambda : |\lambda| = a} \mathbb{S}_\lambda V,$$

then for any $g \in \mathbb{S}_\lambda V$ and $h \in \mathbb{S}_\mu V$, with $|\lambda| = a$ and $|\mu| = b$, the product (provided $N^{\lambda,\mu}_\nu \neq 0$) $\mathcal{M}^{\lambda,\mu}_\nu (g \otimes h)$ lies in $\mathbb{S}_\nu V$ with $|\nu| = a + b$. In other words,

$$(\mathbb{S}^\bullet V)_a \cdot (\mathbb{S}^\bullet V)_b \subset (\mathbb{S}^\bullet V)_{a+b}.$$

2.3.2 The Schur Apolarity Action

We define the *Schur apolarity action* (cf. [S18], p. 47) as follows. Suppose

$$\mathbb{S}_\lambda V \subset \bigotimes_i \wedge^{\lambda'_i} V \quad \text{and} \quad \mathbb{S}_\mu V^* \subset \bigotimes_i \wedge^{\mu'_i} V^*,$$

where λ' and μ' denote the conjugate partitions. Then the apolarity action is given by the tensor product of the standard skew–apolarity actions:

$$\varphi \colon \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \bigotimes_i \wedge^{\lambda'_i - \mu'_i} V.$$

If $\mu \not\subset \lambda$, we define φ to be zero. In the case $\mu \subset \lambda$, the image of φ is in fact contained in the skew Schur module $\mathbb{S}_{\lambda/\mu} V$.

Given $f \in \mathbb{S}_\lambda V$ and a partition $\mu \subset \lambda$, one defines the *catalecticant map*

$$\mathcal{C}_f^{\lambda,\mu} \colon \mathbb{S}_\mu V^* \to \mathbb{S}_{\lambda/\mu} V, \quad g \mapsto \varphi(f \otimes g).$$

The *apolar set* of f is then defined by

$$f^\perp := \bigoplus_\mu \ker \mathcal{C}_f^{\lambda,\mu} \subset \mathbb{S}^\bullet V^*.$$

Definition 2.28. *Let $\lambda = (\lambda_1^{a_1}, \dots, \lambda_k^{a_k})$ be a partition, such that $a_1 + \cdots + a_k < n := \dim(V)$. Consider the flag variety $\mathbb{F}(n_1, \dots, n_k; V)$ as defined in section 2.1.2. A point $p \in X$ is of the form*

$$p = (v_1 \wedge \cdots \wedge v_{n_k})^{\otimes d_k} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1},$$

representing the flag

$$V_1 = \langle v_1, \dots, v_{n_1} \rangle \subset \cdots \subset V_k = \langle v_1, \dots, v_{n_k} \rangle.$$

Their annihilators are generated by

$$V_1^\perp = \langle x_{n_1+1}, \dots, x_n \rangle \supset \cdots \supset V_k^\perp = \langle x_{n_k+1}, \dots, x_n \rangle.$$

Now, consider

$$\mathrm{Sym}^1 V_k^\perp,\ \mathrm{Sym}^{d_k+1} V_{k-1}^\perp, \dots,\ \mathrm{Sym}^{d_k+\cdots+d_2+1} V_1^\perp$$

which we call "generators". We define the ideal $I(p)$ associated with the point p as the ideal generated by the generators inside the graded ring $\left(\mathbb{S}^\bullet V^*,\ \mathcal{M}_\nu^{\lambda,\mu}\right)$.

With all the above, we are able to recover the analogue of the Apolarity lemma that works for tensors arising from Schur modules.

Theorem 2.29 (Schur Apolarity Lemma, cf. ([S18], p. 57))**.** *Let λ be a partition with $l(\lambda) < \dim V$ and let $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ be the minimal $SL(V)$–orbit. For points $p_1, \dots, p_r \in X$ and for any $f \in \mathbb{S}_\lambda V$, the following statements are equivalent:*

1. *f lies in the linear span $\langle p_1, \dots, p_r \rangle$.*
2. *The ideal*

   $$I(p_1, \dots, p_r) = \bigcap_{i=1}^r I(p_i)$$

   is contained in $f^\perp$.

2.3.3 Examples

We conclude this review by providing two examples of Schur apolarity action.

Example 2.30. Let us consider the partitions $\lambda = (2,2,1)$ and $\mu = (2)$. Since $\mu \subset \lambda$, the Schur apolarity map

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \to \mathbb{S}_{\lambda/\mu} V$$

is non-trivial. Take the elements

$$t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \in \mathbb{S}_\lambda V, \quad s = \alpha \otimes \alpha \in \mathbb{S}_\mu V.$$

The action of Schur apolarity is given by the tensor product of the skew-apolarity maps

$$\varphi_1 : \bigwedge^3 V \otimes V^* \to \bigwedge^2 V, \qquad \varphi_2 : \bigwedge^2 V \otimes V^* \to V.$$

We have:

$$\begin{aligned}
&\varphi_1(v_1 \wedge v_2 \wedge v_3 \otimes \alpha) = \alpha(v_1) \cdot v_2 \wedge v_3 - \alpha(v_2) \cdot v_1 \wedge v_3 + \alpha(v_3) \cdot v_1 \wedge v_2;\\
&\varphi_2(v_1 \wedge v_2 \otimes \alpha) = \alpha(v_1) \cdot v_2 - \alpha(v_2) \cdot v_1.
\end{aligned}$$

Overall:

$$\begin{aligned}
\varphi(t \otimes s) &= \varphi_1(v_1 \wedge v_2 \wedge v_3 \otimes \alpha) \otimes \varphi_2(v_1 \wedge v_2 \otimes \alpha)\\
&= \alpha(v_1)^2 \cdot v_2 \wedge v_3 \otimes v_2 - \alpha(v_1)\alpha(v_2) \cdot v_2 \wedge v_3 \otimes v_1\\
&\quad - \alpha(v_1)\alpha(v_2) \cdot v_1 \wedge v_3 \otimes v_2 + \alpha(v_2)^2 \cdot v_1 \wedge v_3 \otimes v_1\\
&\quad + \alpha(v_1)\alpha(v_3) \cdot v_1 \wedge v_2 \otimes v_2 - \alpha(v_2)\alpha(v_3) \cdot v_1 \wedge v_2 \otimes v_1.
\end{aligned}$$

Example 2.31. Let us consider $\lambda = (2,2,2)$ and $\mu = (1,1,1)$. Again, since $\mu \subset \lambda$, the Schur apolarity map

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \to \mathbb{S}_{\lambda/\mu} V$$

is non-trivial. Define the tensor

$$t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_3 \wedge v_4 + v_1 \wedge v_3 \wedge v_4 \otimes v_1 \wedge v_2 \wedge v_3 \in \mathbb{S}_\lambda V,$$

and let

$$s = x_1 \wedge x_2 \wedge x_3 \in \mathbb{S}_\mu V^*,$$

where $x_i(v_j) = \delta_{ij}$. Then, through Schur apolarity, we find

$$\begin{aligned}
\varphi(t \otimes s) &= \det \begin{pmatrix} x_1(v_1) & x_1(v_2) & x_1(v_3) \\ x_2(v_1) & x_2(v_2) & x_2(v_3) \\ x_3(v_1) & x_3(v_2) & x_3(v_3) \end{pmatrix} \cdot v_1 \wedge v_3 \wedge v_4\\
&\quad + \det \begin{pmatrix} x_1(v_1) & x_1(v_3) & x_1(v_4) \\ x_2(v_1) & x_2(v_3) & x_2(v_4) \\ x_3(v_1) & x_3(v_3) & x_3(v_4) \end{pmatrix} \cdot v_1 \wedge v_2 \wedge v_3\\
&= v_1 \wedge v_3 \wedge v_4.
\end{aligned}$$

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