# Apolarity for border cactus decomposition in case of Veronese embedding

Weronika Buczyńska Jarosław Buczyński

22nd July 2020

**Abstract**

We present the border apolarity for cactus varieties to Veronese varieties. The border apolarity technique was introduced in our earlier work for secant varieties to any smooth toric projective varieties. These working notes are provided as a temporary reference for other authors, while we are writting the details of the generalisation to any toric variety.

**addresses:**
J. Buczyński, `jabu@mimuw.edu.pl`, Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland
W. Buczyńska, `wkrych@mimuw.edu.pl`, Faculty of Mathematics, Computer Science and Mechanics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

## 1 Introduction

Let $\Bbbk$ be any algebraically closed base field. Fix a finite dimensional vector space $V$ over $\Bbbk$, and pick a finite subscheme $R \subset \mathbb{P}V$ of length $R$. Assume in addition that $R$ is in a linearly nondegererate position, that is, its linear span $\langle R \rangle$ has maximal possible dimension: $\min\left\{r, \dim S^d V\right\}$. Let $I(R) \subset S^\bullet V^*$ be its homogeneous ideal, and let $h : \mathbb{Z} \to \mathbb{N}$ be the Hilbert function of $I$, $h(d) := \dim(S^\bullet V^*/I(R))_d$.

For every non-negative integer $r$ we define an integer valued function $h_r \colon \mathbb{Z} \to \mathbb{N}$:

$$h_r(i) = \min(r, \dim S^i V^*).$$

Then it is well known that the function $h$ as above satisfies the following conditions:

(i) $h(1) = \min\{r, \dim V\} = h_r(1)$,

(ii) $h(d) \leq h_{\mathcal{H}}(d+1)$,

(iii) if $h(d) = h(d+1)$, then $h(e) = r$ for all $e \geq d$,

(iv) $h(d+1) \leq h(d)^{\langle d \rangle}$,

(v) $h(d) \leq h_r(d)$ for all $d$.

Denote by $S^d V$ the space of divided power polynomials of degree $d$ in $\dim V$-variables and coefficients in $\Bbbk$. Consider the Veronese embedding $v_d \colon \mathbb{P}V \to \mathbb{P}(S^d V)$. For $W \subset S^d V$ a linear subspace of dimension $i$ define The *cactus rank* of $W$ to be the minimal number $r = \operatorname{cr}(W)$ such that $W \subset \langle v_d(R) \rangle$ for a subscheme $R \subset \mathbb{P}V$ of length $r$. The *Grassmann cactus variety* is the following Zariski closure in the Grassmannian $Gr(i, S^d V)$:

$$\mathfrak{K}_{r,i}\left(v_d(\mathbb{P}V)\right) := \overline{\bigcup \{W \in Gr(i, S^d V) \mid \operatorname{cr}(W) \leq r\}}.$$

The *border cactus rank* $\operatorname{bcr}(W)$ of $W$ is the minimal number $r$ such that $W \in \mathfrak{K}_{r,i}\left(v_d(\mathbb{P}V)\right)$.

In this note we present the proof of the following characterisations of the border cactus rank.

**Theorem 1.1** (Weak Apolarity for Border Cactus Decompositions, or weak ABCD)**.** *Suppose $W \subset S^d V$. If $\operatorname{bcr}(W) \leq r$ then there exists a homogeneous ideal $I \subset S^\bullet V^*$ with Hilbert function $h$ satisfying Properties (i)–(v), such that $I \subset \operatorname{Ann}(W)$. Moreover if:*

- $\operatorname{char}(\Bbbk) \neq 2, 3$ *and* $r \leq 8$, *or*
- $\dim W = 1$ *and* $r \leq 14$ *and either:*
  - $\operatorname{char}(\Bbbk) = 0$ *or*
  - $\operatorname{char}(\Bbbk) \neq 2, 3$ *and* $\dim V \leq 6$,

*then $I$ can be chosen to have Hilbert function $h_r$.*

The above theorem is a consequence of stronger Theorem 1.3, which is an "if and only if" statement. The argument is presented in Section 6. we phrase below. For this purpose, we compare the multigraded Hilbert scheme $\operatorname{Hilb}(S^\bullet V^*)$ and the usual Hilbert scheme $\mathcal{H}ilb(\mathbb{P}V)$.

**Proposition 1.2.** *For each irreducible component $\mathcal{H} \subset \mathcal{H}ilb(\mathbb{P}V)$ there is a unique irreducible component $\mathbf{H}_{\mathcal{H}} \subset \operatorname{Hilb}(S^\bullet V^*)$ such that:*

- *a general ideal $I \in \mathbf{H}_{\mathcal{H}}$ is saturated,*
- *the natural map* $\operatorname{sat}\colon \operatorname{Hilb}(S^\bullet V^*) \to \mathcal{H}ilb(\mathbb{P}V)$ *taking a homogeneous ideal $I \subset S^\bullet V^*$ to the subscheme of projective space defined by $I$ restricts to a birational morphism of reduced subschemes $(\mathbf{H}_{\mathcal{H}})_{\text{red}} \to \mathcal{H}_{\text{red}}$.*

The proof is in Section 5.

For each $\mathcal{H}$ as in the proposition, there is an integer valued function $h_{\mathcal{H}} \colon \mathbb{Z} \to \mathbb{N}$ which is the Hilbert function of any $I \in \mathbf{H}_{\mathcal{H}}$. Note that if $\mathcal{H}$ is a component of the Hilbert scheme of points of length $r$, then $h_{\mathcal{H}}$ satisfies the properties (i)–(v).

**Theorem 1.3** (Apolarity for Border Cactus Decompositions, or ABCD)**.** *Suppose $W \subset S^d V$. Then $\operatorname{bcr}(W) \leq r$ if and only if for some irreducible component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$ of the Hilbert scheme of points of length $r$ there exists a homogeneous ideal $I \subset S^\bullet V^*$ with $I \in \mathbf{H}_{\mathcal{H}}$ and $I \subset \operatorname{Ann}(W)$. Moreover, if $\dim W = 1$, then $\mathcal{H}$ might be chosen to be a component containing Gorenstein schemes.*

The proof is in Section 6.

## 2 Binomial coefficients as a polynomial

Let $d, n$ be two integers with $n \geq 0$. We consider the usual binomial coefficients $\binom{d}{n}$ of integer with the following standard convention:

$$\binom{d}{n} = \begin{cases} \frac{d!}{n!(d-n)!} & \text{if } d \geq n, \\ 0 & \text{if } d < n. \end{cases}$$

We also define a polynomial $\binom{t+d}{n} \in \mathbb{Q}[t]$ in the usual way:

$$\binom{t+d}{n} := \frac{(t+d)(t+d-1)\cdots(t+d-n+1)}{n!}.$$

That is, if $S = S^{\bullet}V^* = \Bbbk[x_0, \dots, x_n]$ with the standard grading, then the value of the Hilbert function of $S(d)$ at $i$ is $\binom{i+d+n}{n}$, while its Hilbert polynomial is $\binom{t+d+n}{n}$.

**Lemma 2.1.** *The value of the polynomial $\binom{t+d}{n}$ at $t_0 \in \mathbb{Z}$ agrees with $\binom{t_0+d}{n}$ if and only if $t_0 \geq -d$.*

*Proof.* If $t_0 \geq n-d$, then the two definition clearly agree. If $-d \leq t_0 < n-d$, then one of $(t+d), (t+d-1), \dots, (t+d-n+1)$ vanishes at $t_0$, hence again the two definitions agree. Finally, if $t_0 < -d$, then the polynomial does not vanish (we have already seen all the $n$ roots), thus it is different than the integer binomial coefficeint. □

**Lemma 2.2.** *Let $M$ be a finitely generated graded module over $S$ and let $D$ be an integer. If the only summands in the terms of the minimal free resolution of $M$ are of the form $S(d)$ for some $d \geq -D-n$, then the value of Hilbert polynomial of $M$ at $t_0$ agrees with the value of Hilbert function of $M$ at $t_0$ for all $t_0 \geq D$ .*

*Proof.* The value of Hilbert function of $M$ is calculated by addding (with signs) the Hilbert functions of $S(d)$ appearing in the minimal free resolution, while its polynomial is obtained in the same way from the Hilbert polynomials of $S(d)$. Since the Hilbert polynomial of $S(d)$ is equal to $\binom{t+d+n}{n}$, and $t_0 \geq D \geq -d-n$, the claim follows from Lemma 2.1. □

**Lemma 2.3.** *Suppose $M$ is a $D$-regular graded $S$ module. Then the value of Hilbert polynomial of $M$ at $t_0$ agrees with the value of Hilbert function of $M$ at $t_0$ for all $t_0 \geq D$.*

*Proof.* By Eisenbud-Goto Theorem [BH93, Thm 4.3.1(a),(c)] $M_{\geq D}$ has a linear minimal free resolution. By Hilbert Syzygy Theorem [Eise95, Thm 1.13] the length of the minimal free resolution is at most $n+1$, and by its linearity satisifies the assumption of Lemma 2.2. Therefore, the value of Hilbert polynomial of $M$ at $t_0$ agrees with the value of Hilbert function of $M$ at $t_0$ for all $t_0 \geq D$, as claimed. □

Now we work with homogeneous ideals $I \subset S$ as the graded modules over $S$. Let $P(I)$ be the Hilbert polynomial of $S/I$. Gotzmann regularity theorem [Gree98, Thm 3.11]:

$$P(I)(t) = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \cdots + \binom{t+a_s-(s-1)}{a_s}, \tag{2.4}$$

where $a_1 \geq a_2 \geq \cdots \geq a_s \geq 0$. Moreover, if $I$ is saturated, then $I$ is $s$ regular. Nevertheless, note that:

(a) All saturated homogeneous ideals with Hilbert polynomial $P(I)$ are $s$-regular.

(b) Some saturated homogeneous ideals with Hilbert polynomial $P(I)$ could be $D$-regular for $D < s$.

(c) Some non-saturated homogeneous ideals with Hilbert polynomial $P(I)$ could be $D$-regular only starting from some $D > s$.

**Lemma 2.5.** *Suppose a homogeneous ideal $I \subset S$ is $D$-regular and $s, a_s$ are defined as above. Then the growth of $I$ from $I_d$ to $I_{d+1}$ is minimal possible for all $d \geq \max(D, s-1-a_s)$, that is:*

$$\dim(S/I)_{d+1} = (\dim(S/I)_d)^{\langle d \rangle} .$$

*Proof.* Write the Hilbert polynomial $P(I)$ of $S/I$ in the binomial presentation as in (2.4). If $d \geq s-1-a_s$, then also $d \geq i-1-a_i$ for all $i$. Therefore $P(I)(d) = \binom{d+a_1}{d} + \binom{d+a_2-1}{d-1} + \cdots + \binom{d+a_s-(s-1)}{d-s+1}$ by Lemma 2.1. Thus for all such $d$ we have $P(I)(d+1) = P(I)(d)^{\langle d \rangle}$.

On the other hand, if $d \geq D$, then we also have $h(I)(d) = P(I)(d)$ by Lemma 2.2. Since the ideal is generated in degrees at most $D$ by Eisenbud-Goto Theorem [BH93, Thm 4.3.1(a),(c)] the claim follows from Gotzmann persistence theorem. □

# 3 Saturation is an open property

For a homogeneous ideal $I \subset S^\bullet V^*$, let:

- $Z(I) \subset \mathbb{P}(V)$ be the scheme defined by $I$,
- $\mathcal{I}_{Z(I)} \subset \mathcal{O}_{\mathbb{P}(V)}$ be the ideal sheaf of $Z(I)$,
- $I^{\mathrm{sat}}$ be the saturation of $I$,
- $h^{\mathrm{sat}}(I)$ be the Hilbert function of $S^\bullet V^*/I^{\mathrm{sat}}$, so that

$$h^{\mathrm{sat}}(I)(d) = \binom{n+d}{d} - \dim H^0(\mathcal{I}_{Z(I)}(d)),$$

- $P(I)$ be the Hilbert polynomial of $S^\bullet V^*/I$, so that $P(I)(d) = h^{\mathrm{sat}}(I)(d)$ for large values of $d$.

**Lemma 3.1.** *A homogeneous ideal $I \subset S^\bullet V^*$ is saturated if and only if $\dim I_d = \dim H^0(\mathcal{I}_{Z(I)}(d))$ for all integers $d$.*

**Proof.** Since $H^0(\mathcal{I}_{Z(I)}(d)) = (I^{\mathrm{sat}})_d$ and $I \subset I^{\mathrm{sat}}$ we have

$$I = I^{\mathrm{sat}} \iff \forall_d I_d = H^0(\mathcal{I}_{Z(I)}(d)) \iff \forall_d \dim I_d = \dim H^0(\mathcal{I}_{Z(I)}(d)).$$

□

**Lemma 3.2.** *Let $P$ be an integer valued polynomial. Then there exists a finite number $D$ such that for all homogeneous ideals $I \subset S^\bullet V^*$ with $P(I) = P$ the ideal $I$ is saturated if and only if $\dim I_d = \dim H^0(\mathcal{I}_{Z(I)}(d))$ for all integers $0 \leq d \leq D$.*

*Proof.* If there is no homogeneous ideal with Hilbert polynomial $P$, then there is nothing to prove. Otherwise, let $D$ be the Gotzmann's regularity of $P$, so that all saturated homogeneous ideals with Hilbert polynomial $P$ are $D$-regular [BH93, Thm 4.3.2]. If $I$ is saturated, then the claim follows from Lemma 3.1.

Conversely, suppose $\dim I_d = \dim H^0(\mathcal{I}_{Z(I)}(d))$ for all integers $0 \le d \le D$. Then $(I^{\mathrm{sat}})_d = I_d$ for all $d \le D$. Since $I^{\mathrm{sat}}$ is $D$-regular, By [BH93, Thm 4.3.1(a), (c)] the minimal generators of $I^{\mathrm{sat}}$ only appear in degrees at most $D$. In particular, $I$ has no minimal generators in degree $D$. Also, by Lemma 2.5, the saturated ideal $I^{\mathrm{sat}}$ has a minimal growth for all $d \ge D$. Thus, by induction starting from $d = D$, and Macaulay growth theorem we show that $\dim I_d = \dim(I^{\mathrm{sat}})_d$ for all $d \ge D$:

$$\begin{aligned}\dim(S^\bullet V^*/I^{\mathrm{sat}})_{d+1} &\le \dim\left(S^\bullet V^*/I\right)_{d+1} \le \left(\dim(S^\bullet V^*/I)_d\right)^{\langle d\rangle} \\ &= \left(\dim(S^\bullet V^*/I^{\mathrm{sat}})_d\right)^{\langle d\rangle} = \dim(S^\bullet V^*/I^{\mathrm{sat}})_{d+1}.\end{aligned}$$

This concludes the proof by Lemma 3.1. □

Fix $h\colon \mathbb{Z} \to \mathbb{N}$, an integer valued function and a vector space $V$. Consider the multigraded Hilbert scheme $\mathrm{Hilb}_h S^\bullet V^*$ of homogeneous ideals in $S^\bullet V^*$ with Hilbert function $h$. The main result of this section is:

**Proposition 3.3.** *The subset of* $\mathrm{Hilb}_h S^\bullet V^*$ *consisting of saturated ideals is Zariski open.*

**Proof.** We fix an irreducible component $\mathbf{H} \subset (\mathrm{Hilb}_h S^\bullet V^*)_{\mathrm{red}}$ (in particular, $\mathbf{H}$ is reduced and connected). It is enough to prove that the set of saturated ideals in $\mathbf{H}$ is Zariski open in $\mathbf{H}$. If there is no saturated ideal in $\mathbf{H}$, then the set is empty, in particular open. Thus suppose $I_0 \in \mathbf{H}$ is a saturated ideal, and let $P$ be the Hilbert polynomial of $Z(I_0)$. Note that $P$ is the Hilbert polynomial of every ideal in $\mathbf{H}$, as all ideals have the same Hilbert function and the Hilbert polynomial is determined by the Hilbert function.

Let $\mathcal{J} \subset \mathcal{O}_{\mathbf{H}} \otimes S^\bullet V^*$ be the universal (homogeneous) ideal sheaf arising from the definition of the multigraded Hilbert scheme. That is, for each $I \in \mathbf{H}$, if $\mathfrak{m}_I \subset \mathcal{O}_{\mathbf{H}}$ is the maximal ideal of $I$, then the ideal $I \subset S^\bullet V^*$ is equal to

$$\mathcal{J} \otimes_{\mathcal{O}_{\mathbf{H}}} \mathcal{O}_{\mathbf{H}}/\mathfrak{m}_I \subset \mathcal{O}_{\mathbf{H}}/\mathfrak{m}_I \otimes S^\bullet V^* = S^\bullet V^*.$$

Then $\mathcal{J}$ defines a subscheme $\mathcal{R} \subset \mathbf{H} \times \mathbb{P}V$, which is flat over $\mathbf{H}$ by [Hart77, Thm III.9.9]. Denote by $\mathcal{I}_{\mathcal{R}} \subset \mathcal{O}_{\mathbf{H}\times\mathbb{P}V} = \mathcal{O}_{\mathbf{H}} \otimes \mathcal{O}_{\mathbb{P}V}$ the ideal sheaf of $\mathcal{R}$. For each $I \in \mathbf{H}$ we can recover $I^{\mathrm{sat}}$ from $\mathcal{I}_{\mathcal{R}}$:

$$(I^{\mathrm{sat}})_d = H^0(\mathcal{I}_{\mathcal{R}}(d) \otimes_{\mathcal{O}_{\mathbf{H}}} (\mathcal{O}_{\mathbf{H}}/\mathfrak{m}_I))$$

for any integer $d$. But the dimension of the right hand side is upper semicontinuous in $I$ by [Hart77, Thm III.12.8]. That is, for each integer $d$, the subset $W_d \subset \mathbf{H}$ defined as:

$$W_d = \left\{ I \in \mathbf{H} \middle| \dim(S^\bullet V^*/I^{\mathrm{sat}})_d < h(d) \right\}$$

is Zariski closed.

To complete the proof we use Lemma 3.2, by which the set of saturated ideals in $\mathbf{H}$ is the complement of the union of $W_d$ for all finitely many values of $d$. In particular, this set is open. □

# 4 Generic Hilbert functions

Consider the ordinary Hilbert scheme $\mathcal{H}ilb(\mathbb{P}V)$. For each irreducible component $\mathcal{H} \subset \mathrm{Hilb}(\mathbb{P}V)$ define a function $h_{\mathcal{H}}\colon \mathbb{Z} \to \mathbb{N}$ by

$$h_{\mathcal{H}}(d) := \max\left\{\binom{d+n}{n} - \dim I(R)_d \mid R \in \mathcal{H}\right\}.$$

We call $h_{\mathcal{H}}$ the *generic Hilbert function of* $\mathcal{H}$.

**Lemma 4.1.** *The subset of* $\mathcal{H}$ *defined as:*

$$U^{gen}_{\mathcal{H}} := \{R \in \mathcal{H} \mid \forall_{d\in\mathbb{Z}}\ \dim\left(S^{\bullet}V^*/I(R)\right)_d = h_{\mathcal{H}}(d)\}$$

*is Zariski open and dense.*

**Proof.** If $\dim \mathcal{H} = 0$, then the claim is clear.

Otherwise, we mimic the proof of Proposition 3.3. Denote by $\mathcal{R} \subset \mathcal{H} \times \mathbb{P}V$ the universal subscheme, and by $\mathcal{I}_{\mathcal{R}} \subset \mathcal{O}_{\mathcal{H}\times\mathbb{P}V} = \mathcal{O}_{\mathcal{H}} \otimes \mathcal{O}_{\mathbb{P}V}$ the ideal sheaf of $\mathcal{R}$. Note $\mathcal{I}_{\mathcal{R}}$ is flat over $\mathcal{H}$, so also the twist $\mathcal{I}_{\mathcal{R}}(d)$ is flat for all integers $d$. For $R \in \mathcal{H}$, let $\mathfrak{m}_R \subset \mathcal{O}_{\mathcal{H}}$ be the maximal ideal sheaf of $R$.

The semicontinuity theorem [Hart77, Thm III.12.8] implies that

$$\dim H^0\left(\mathcal{I}_{\mathcal{R}}(d) \otimes_{\mathcal{O}_{\mathcal{H}}} (\mathcal{O}_{\mathcal{H}}/\mathfrak{m}_R)\right) = \dim I(R)_d$$

is upper semicontinuous in $R$. Therefore, for each integer $d$, the subset $W_d \subset \mathcal{H}$ defined as:

$$W_d = \{R \in \mathcal{H} |\dim(S^{\bullet}V^*/I(R))_d < h_{\mathcal{H}}(d)\}$$

is Zariski closed. Moreover, $W_d \neq \mathcal{H}$ by the definition of $h_{\mathcal{H}}$.

Finally, by Lemma 2.5 there is an integer $D$, such that the union of $W_d$ is equal to $\bigcup_{d=0}^{D} W_D$, thus it is also Zariski closed and not equal to $\mathcal{H}$, since $\dim \mathcal{H} > 0$. By definition, $U^{gen}_{\mathcal{H}}$ is the complement of the union of all $W_d$, thus it is open and dense, as claimed. □

**Lemma 4.2.** *Suppose* $i_1\colon R \hookrightarrow \mathbb{P}V$ *is an embedding of a finite scheme of length* $r$*. Then there is an embedding* $i_2\colon R \hookrightarrow \mathbb{P}V$ *such that* $\dim(V^*/I(i_2(R))_1) = \min\{r, \dim V\}$.

*Proof.* First there is always a concisely independent embedding $i_3\colon R \hookrightarrow \mathbb{P}^{r-1}$ with $I(i_3(R))_1 = 0$ [BBKT15, pp 702–703]. This is obtained by the embedding $R = \operatorname{Spec} A$ into $\mathbb{P}A$, or equivalently, by the trivial line bundle $\mathcal{O}_R$, which is very ample. If $\dim V \geq r$, then $i_2$ is a composition of $i_3$ with a linear embedding $\mathbb{P}^{r-1} \subset \mathbb{P}V$.

Suppose $\dim V < r$. Let $\mathfrak{K}_2\left(i_2(R)\right) \subset \mathbb{P}^{r-1}$ be the second cactus variety of $R$, that is the finite union of the (projective) Zariski tangent spaces of $R$ at each point and the secant lines connecting the any two points of support of $R$. Note that $\dim \mathfrak{K}_2\left(i_3(R)\right) \leq \dim \mathbb{P}V$: indeed, since $i_1$ is an embedding of $R$ into $\mathbb{P}V$, each tangent space must be at most $\dim \mathbb{P}V$ dimensional. Also each secant line is one dimensional, and there are any secant lines only if there are at least two distinct points of support, which is possible only if $\dim \mathbb{P}V \geq 1$.

Pick a linear projection $\mathbb{P}^{r-1} \dashrightarrow \mathbb{P}V$. By standard arguments, such as in [Hart77, Prop. IV.3.4], if the center of the projection does not intersect $\mathfrak{K}_2\left(i_3(R)\right)$, then the composition of $i_3$ and the projection is still an embedding. Moreover, in such a case the linear span of this new embedding is $\mathbb{P}V$. Clearly, by the dimension count, we can pick the linear projection satisfying the above property. □

**Lemma 4.3.** *Suppose $i_1$ and $i_2$ are two embeddings of the same finite scheme $R \hookrightarrow \mathbb{P}V$. Then $i_1(R)$ and $i_2(R)$ are in the same component of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}V)$, where* $r = \operatorname{length} R$

*Proof.* Pick a basis $\{\alpha_0, \ldots, \alpha_n\}$ of $V^* = H^0(\mathcal{O}_{\mathbb{P}V}(1))$. Pulling the basis back to $R$ using $i_1$ and $i_2$, we obtain two collections, $\{i_1^*\alpha_0, \ldots, i_1^*\alpha_n\}$ and $\{i_2^*\alpha_0, \ldots, i_2^*\alpha_n\}$, of sections of the trivial line bundle $\mathcal{O}_R \simeq i_1^*\mathcal{O}_{\mathbb{P}V}(1) \simeq\simeq i_2^*\mathcal{O}_{\mathbb{P}V}(1)$. Consider the $\mathbb{P}^1$-parametrised family of maps $R \to \mathbb{P}V$ determined by

$$\{s \cdot i_1^*\alpha_0 + t \cdot i_2^*\alpha_0, \ldots, s \cdot i_1^*\alpha_n + t \cdot i_2^*\alpha_n\},$$

where $s, t$ are coordinated on $\mathbb{P}^1$. Generically, this is an embedding, and thus we obtain a flat family of subschemes of $\mathbb{P}V$ paramerised by an open dense subset of $\mathbb{P}^1$. This family demonstrates that $i_1$ and $i_2$ are in the same component of the Hilbert scheme, as claimed. □

**Proposition 4.4.** *Suppose $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$ is an irreducible component of the Hilbert scheme of finite subschemes of length $r$. Then $h_{\mathcal{H}}$ satisifies the properties (i)–(v) of page 1.*

*Proof.* Items (ii), (iii), (iv), (v) follow from Lemma 4.1, since $h_{\mathcal{H}}$ is a Hilbert function of a saturated ideal of a finite scheme of length $r$.

To see that (i) holds, take a general element $R \in \mathcal{H}$. By the generality, $\mathcal{H}$ is the unique component of $\mathcal{H}ilb_r(\mathbb{P}V)$ containing $R$. By Lemma 4.2, we can reembed $R$ into $\mathbb{P}V$ in a linearly nondegererate way. By Lemma 4.3 both embeddings are in the same component, that is $\mathcal{H}$. It follows that $h_{\mathcal{H}}(1) \geq \min\{r, \dim V\}$, then by (v) we must have the equality. □

*Example* 4.5. If $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$ is the smoothable component, then $h_{\mathcal{H}} = h_r$.

*Proof.* Since the base field $\Bbbk$ is algebraically closed, it is infinite, hence for each $d$ there is an embedding of a collection of $r$ distinct points $R$, such that $v_d(R)$ is in linearly general position. □

*Example* 4.6. If $r \leq 8$ and $\operatorname{char}(\Bbbk) \neq 2, 3$, then $h_{\mathcal{H}} = h_r$ for any component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$

*Proof.* By [CEVV09], either $\mathcal{H}$ is the smoothable component, or it is the $(1, 4, 3)$ component. In the first case, the claim follows from Example 4.5.

In the latter case, we must have $\dim V \geq 5$, $r = 8$, and the natural embedding of any $(1, 4, 3)$-scheme into a $\mathbb{P}^4$ shows $h_{\mathcal{H}}(2) \geq 8$. We conclude applying Items (i), (ii), and (v) of Proposition 4.4. □

*Example* 4.7. Suppose $\operatorname{char}(\Bbbk) \neq 2, 3$. If $\dim V \geq 7$ and $\mathcal{H} \subset \mathcal{H}ilb_14(\mathbb{P}V)$ is the closure of the (1661)-component in the Gorenstein locus (that is, the component that contains all finite Gorenstein schemes of length 14 with the local Hilbert function (1661)), then $h_{\mathcal{H}} = h_r$.

*Proof.* If $\dim V = 7$, then $\mathcal{H}$ contains $\operatorname{Proj} S^\bullet V^*/(I^{hom})$, where $I^{hom}$ is the homogenisation of the ideal $I \subset \Bbbk[x_1, \ldots, x_6]$ obtained as $I = \operatorname{Ann}(x_1^3 + \cdots + x_6^3 + x_1 x_2)$. $I^{hom}$ is saturated, and it is straightforward to verify (for instance on a computer algebra system), that $\dim(I^h om)_2 = 14$, giving that $h_{\mathcal{H}}(2) \geq 28 - 14 = 14$.

If $\dim V > 7$ we can also use the same embedding into some linearly embedded $\mathbb{P}^6 \subset \mathbb{P}V$, to show that $h_{\mathcal{H}}(2) \geq 14$ in general. Then Items (i), (ii), and (v) of Proposition 4.4 guarantee that $h_{\mathcal{H}} = h_r$. □

# 5 The ideal to scheme and linear span morphisms

The natural morphism $\operatorname{sat}\colon \operatorname{Hilb}(S^\bullet V^*) \to \mathcal{H}ilb(\mathbb{P}V)$ is defined by sending a homogeneous ideal $I$ to $\operatorname{Proj}(S^\bullet V^*/I)$. Strictly speaking, the universal ideal $\mathcal{I} \subset \mathcal{O}_{\operatorname{Hilb}(S^\bullet V^*)} \otimes S^\bullet V^*$ defines a subscheme $\mathcal{R} \subset \operatorname{Hilb}(S^\bullet V^*) \times \mathbb{P}V$, which is flat over $\operatorname{Hilb}(S^\bullet V^*)$ by [Hart77, Prop. III.9.9] or [FGI+05, Lem. 5.5] or [Vaki17, Exercise 24.7.A(b)] (one needs to argue for each Hilbert function separately).

Conversely, if $\mathcal{H} \subset \mathcal{H}ilb(\mathbb{P}V)$ is any irreducible component, and $U_{\mathcal{H}}^{gen} \subset \mathcal{H}$ is the open dense subset of $\mathcal{H}$ as in Proposition 4.4, then we obtain an inverse map $U_{\mathcal{H}}^{gen} \to \operatorname{Hilb}_{h_{\mathcal{H}}}(S^\bullet V^*)$. For the purpose of this note, it is enough to construct the map from reduced part of $U_{\mathcal{H}}^{gen}$, and we will restrict to this case:

**Lemma 5.1.** *Let $\mathcal{H}$ be the reduced subscheme of an irreducible component of $\mathcal{H}ilb(\mathbb{P}V)$, $U_{\mathcal{H}}^{gen}$ be its open subset of subschemes with ideal generic Hilbert function, $\mathcal{R} \subset U_{\mathcal{H}}^{gen} \times \mathbb{P}V$ be the universal subscheme, and $\mathcal{I}_{\mathcal{R}} \subset \mathcal{O}_{U_{\mathcal{H}}^{gen} \times \mathbb{P}V}$ be the ideal sheaf of $\mathcal{R}$. Denote by $\pi\colon U_{\mathcal{H}}^{gen} \times \mathbb{P}V \to U_{\mathcal{H}}^{gen}$ the projection. Then the sheaf of homogeneous ideals $\bigoplus_{i=0}^{\infty} \pi_*(\mathcal{I}_{\mathcal{R}}(d))$ is flat and determines a map* $\operatorname{ideal}\colon U_{\mathcal{H}}^{gen} \to \operatorname{Hilb}_{h_{\mathcal{H}}}(S^\bullet V^*)$, *such that* $\operatorname{sat} \circ \operatorname{ideal} = \operatorname{id}_{U_{\mathcal{H}}^{gen}}$. *Moreover,* ideal *is an open immersion to the reduced subscheme of* $\operatorname{Hilb}_{h_{\mathcal{H}}}(S^\bullet V^*)$.

*Proof.* For every $d$ and for each point $R \in U_{\mathcal{H}}^{gen}$ the dimension of the fibre $\pi_*(\mathcal{I}_{\mathcal{R}}(d))_R$ (as a $k(R)$-vector space) is constant (independent of $R$) and equal to $h_{\mathcal{H}}(d)$. Thus by Nakayama's Lemmathe sheaf $\pi_*(\mathcal{I}_{\mathcal{R}}(d))$ is locally free, hence flat. Therefore also $\bigoplus_{i=0}^{\infty} \pi_*(\mathcal{I}_{\mathcal{R}}(d))$ is flat, and the universal property of $\operatorname{Hilb}_{h_{\mathcal{H}}}$ gives us the desired map $\operatorname{ideal}\colon U_{\mathcal{H}}^{gen} \to \operatorname{Hilb}_{h_{\mathcal{H}}}(S^\bullet V^*)$.

On points, the map ideal takes a subscheme $R$ to its (saturated) homogeneous ideal $I(R) \subset S^\bullet V^* \otimes k(R)$, then the map sat takes $I(R)$ to the scheme in $\mathbb{P}V$ defined by $I(R)$, that is $R$. That is, the composition $\operatorname{sat} \circ \operatorname{ideal}$ is the identity on points. Therefore, it is the identity, since $U_{\mathcal{H}}^{gen}$ is integral [Hart77, Lem. I.4.1].

Finally, ideal is an open immersion by Proposition 4.4 and the fact that every subscheme of projective space is uniquely determined its saturated ideal. □

Thus for each Hilbert polynomial $P$ we have a bunch of distinguished components of $\operatorname{Hilb}(S^\bullet V^*)$.

**Definition 5.2.** Fix an integer valued polynomial $P$.

- Suppose $\mathcal{H} \subset \mathcal{H}ilb_P(\mathbb{P}V)$ is an irreducible component. Define the *component of* $\operatorname{Hilb}(S^\bullet V^*)$ *corresponding to* $\mathcal{H}$ to be $\mathbf{H}_{\mathcal{H}} := \overline{\operatorname{ideal}(U_{\mathcal{H}}^{gen})}$.

- Define the *components of* Hilb$(S^\bullet V^*)$ *distiguished for* $P$ to be the set

$$\{\mathbf{H}_{\mathcal{H}} \mid \mathcal{H} \subset \mathcal{H}ilb_P(\mathbb{P}V)\}.$$

**Proposition 5.3.** *The saturation map restricted to the union of distinguished components is a birational map:*

$$\mathrm{sat}\colon \bigcup\{\mathbf{H}_{\mathcal{H}} \mid \mathcal{H} \subset \mathcal{H}ilb_P(\mathbb{P}V)\} \to \mathcal{H}ilb_P(\mathbb{P}V)_{\mathrm{red}}.$$

*Proof.* Clear from the definition of $\mathbf{H}_{\mathcal{H}}$ and Lemma 5.1. □

*Proof of Proposition 1.2.* The distinguished component $\mathbf{H}_{\mathcal{H}}$ defined above satisfies the property of the first item in the proposition by the construction of the map ideal and Lemma 5.1. The second item holds by Proposition 5.3.

The uniqueness follows from Proposition 3.3: suppose $\mathbf{H}'$ is another component of the multigraded Hilbert scheme satisfying the two itemised properties. Let $U' \subset \mathbf{H}'$ be the open subset of saturated ideals. Note that $U'$ is not empty by the first item. Then, the birationality from the second item implies that the general element of $\mathcal{H}$ is in $\mathrm{sat}(\mathbf{H}')$. Thus the map ideal maps such general element to $\mathbf{H}'$ and thus $\mathbf{H}' = \mathbf{H}_{\mathcal{H}}$ as claimed. □

The *linear span map* is defined componentwise as:

$$\begin{aligned} \mathrm{lin}_d\colon \mathcal{H} &\dashrightarrow Gr(h_{\mathcal{H}}(d), S^dV) \\ R &\longmapsto \langle v_d(R)\rangle . \end{aligned}$$

**Proposition 5.4.** *The rational map* $\mathrm{lin}_d\colon \mathcal{H} \dashrightarrow Gr(h_{\mathcal{H}}(d), S^dV)$ *is resolved by the pre-composition with the map* sat *restricted to* $\mathbf{H}_{\mathcal{H}}$.

*Proof.* The map sends a general $R$ to $I(R)(d)^\perp$. Thus it agrees generically with the map $I \mapsto I(d)^\perp$, which is a regular map $\mathbf{H}_{\mathcal{H}} \to Gr(h_{\mathcal{H}}(d), S^dV)$. Thus this map is indeed a resolution of $\mathrm{lin}_d$ by Proposition 5.3. □

# 6 Cactus variety

check assumptions of the integers

The *Grassmann-relative linear span* of a component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$, denoted by $\mathfrak{K}_{r,i}(v_d(\mathbb{P}V), \mathcal{H})$, is defined similarly as in [BJ17, §5.6–5.7]. Explicitly, denote by $\mathcal{S} \to Gr(h_{\mathcal{H}}(d), S^dV)$ the universal subbundle, and by $\mathcal{G}r(i, \mathcal{S})$ the Grassmann bundle of $i$-dimensional subspaces of fibres of $\mathcal{S}$. Then define $\mathcal{G} := \mathrm{lin}_d^* \mathcal{G}r(i, \mathcal{S})$, which is a vector bundle over an open dense subset of $\mathcal{H}$. Finally, define $\mathfrak{K}_{r,i}(v_d(\mathbb{P}V), \mathcal{H})$ to be the closure of the image of $\mathcal{G}$ under the projection $\mathcal{G} \to Gr(i, S^dV)$, which is induced from the projection $\mathcal{G}r(i, \mathcal{S}) \to Gr(i, S^dV)$.

The Grassmann cactus variety is the union of $\mathfrak{K}_{r,i}(v_d(\mathbb{P}V), \mathcal{H})$ over all components $\mathcal{H}$ of $\mathcal{H}ilb_r(\mathbb{P}V)$.

**Proposition 6.1.** $\mathfrak{K}_{r,i}(v_d(\mathbb{P}V), \mathcal{H})$ *is equal to the image of*

$$\mathbf{G} := (\mathrm{lin}_d \circ \mathrm{sat})^* \mathcal{G}r(i, \mathcal{S})$$

*under the projection* $\mathbf{G} \to Gr(i, S^dV)$ *induced from the projection* $\mathcal{G}r(i, \mathcal{S}) \to Gr(i, S^dV)$. *In particular, no closure is needed in this definition.*

*Proof.* By Proposition 5.4, the map $\mathrm{lin}_d \circ \mathrm{sat})$ is regular, and its image in the Grassmannian $Gr(h_{\mathcal{H}}(d), S^d V)$ is equal to the closure of the image of $\mathrm{lin}_d$. Thus the closures of the image of the two projections $\mathbf{G} \to Gr(i, S^d V)$ and $\mathcal{G} \to Gr(i, S^d V)$ are equal.

Moreover $\mathbf{G} \to Gr(i, S^d V)$ is a map of projective varieties, thus projective, hence proper, and the image is closed. □

*Proof of Theorem 1.3.* Let $W \subset S^d V$.

First suppose $\mathrm{bcr}(W) \le r$, that is, $W$ is in $\mathfrak{K}_{r,i}(v_d(\mathbb{P}V), \mathcal{H})$ for some irreducible component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$. If $\dim W = 1$, then $\mathcal{H}$ can be chosen to contain Gorenstein schemes by [BB14, Prop. 2.2(ii)] (over $\Bbbk = \mathbb{C}$) or [BJ17, Cor. 6.20] (over any $\Bbbk$). By Proposition 6.1 there exists an ideal $I \in \mathbf{H}_{\mathcal{H}}$, such that $W \subset I_d^{\perp}$. Then $I \subset \mathrm{Ann}(W)$ by standard apolarity arguments, concluding the proof of the first implication of the theorem, and also the "moreover" statement.

Now suppose there exists an irreducible component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$, and an ideal $I \in \mathbf{H}_{\mathcal{H}}$ such that $I \subset \mathrm{Ann}(W)$. In particular, $W \subset I_d^{\perp}$ and $W$ is in the cactus variety as claimed by Proposition 6.1. □

*Proof of Theorem 1.1.* Suppose $W \subset S^d V$ has border cactus rank at most $r$. Then by Theorem 1.3 there exists an ideal $I \in \mathbf{H}_{\mathcal{H}}$ for some irreducible component $\mathcal{H} \subset \mathcal{H}ilb_r(\mathbb{P}V)$, such that $I \subset \mathrm{Ann}(W)$. By the construction of $\mathbf{H}_{\mathcal{H}}$ (Lemma 5.1) the Hilbert function of $I$ is equal to $h_{\mathcal{H}}$, which satisfies properties of (i)–(v) of page 1 by Proposition 4.4.

The cases of rank 8 follows from Example 4.6. To see the case of $\dim W = 1$, by Theorem 1.3, $\mathcal{H}$ above contains Gorenstein schemes. By [CJN15, Thm A], if $\mathrm{char}(\Bbbk) \neq 2, 3$ and $\dim V \le 6$, then $\mathcal{H}$ is the component of smoothable schemes, hence $h_{\mathcal{H}} = h_r$ by Example 4.5. If $\mathrm{char}(\Bbbk) = 0$ and $\dim V \ge 7$, then [CJN15, Thm B] implies that there are two possible components $\mathcal{H}$: the smoothable component (for which we again use Example 4.5) and the (1661) component (for which we use Example 4.7). □

# References


[BB14] Weronika Buczyńska and Jarosław Buczyński. Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes. *J. Algebraic Geom.*, 23:63–90, 2014.

[BBKT15] Weronika Buczyńska, Jarosław Buczyński, Johannes Kleppe, and Zach Teitler. Apolarity and direct sum decomposability of polynomials. *Michigan Math. J.*, 64(4):675–719, 2015.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[BJ17] Jarosław Buczyński and Joachim Jelisiejew. Finite schemes and secant varieties over arbitrary characteristic. *Differential Geom. Appl.*, 55:13–67, 2017.

[CEVV09] Dustin A. Cartwright, Daniel Erman, Mauricio Velasco, and Bianca Viray. Hilbert schemes of 8 points. *Algebra Number Theory*, 3(7):763–795, 2009.

[CJN15] Gianfranco Casnati, Joachim Jelisiejew, and Roberto Notari. Irreducibility of the Gorenstein loci of Hilbert schemes via ray families. *Algebra Number Theory*, 9(7):1525–1570, 2015.

[Eise95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[Gree98] Mark L. Green. Generic initial ideals. In *Six lectures on commutative algebra*, volume 166 of *Progress in Mathematics*, pages 119–186. Birkhäuser Verlag, Basel, 1998.

[Hart77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Vaki17] Ravi Vakil. The rising sea: Foundations of algebraic geometry notes. a book in preparation, November 18, 2017 version, `http://math.stanford.edu/~vakil/216blog`, 2017.