CALCULUS OF VARIATIONS AND PDEs
BANACH CENTER PUBLICATIONS, VOLUME 101
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

HARDY–POINCARÉ TYPE INEQUALITIES DERIVED FROM *p*-HARMONIC PROBLEMS

IWONA SKRZYPCZAK

Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw
ul. Banacha 2, 02–097 Warszawa, Poland
E-mail: iskrzypczak@mimuw.edu.pl

Abstract. We apply general Hardy type inequalities, recently obtained by the author. As a consequence we obtain a family of Hardy–Poincaré inequalities with certain constants, contributing to the question about precise constants in such inequalities posed in [3]. We confirm optimality of some constants obtained in [3] and [8]. Furthermore, we give constants for generalized inequalities with the proof of their optimality.

1. Introduction. In this paper we derive Hardy–Poincaré inequalities having the form

$$C\int_{\mathbb{R}^n}|\xi|^p\big[(1+|x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma-1}\,dx\le\int_{\mathbb{R}^n}|\nabla\xi|^p\big[(1+|x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma}\,dx,\tag{1}$$

where $C>0$, $1<p<\infty$, $\gamma\in\mathbb{R}$, valid for every Lipschitz function ξ with compact support.

The version of this result, when $p=2$,

$$C\int_{\mathbb{R}^n}|\xi|^2(1+|x|^2)^{\gamma-1}\,dx\le\int_{\mathbb{R}^n}|\nabla\xi|^2(1+|x|^2)^{\gamma}\,dx,\tag{2}$$

is of special interest in many disciplines of analysis. Let us recall some applications of (2) to the theory of nonlinear diffusions — evolution equations of a form $u_t=\Delta u^m$, which are called fast diffusion equation (FDE) if $m<1$ and porous media equation (PME) if $m>1$. In the theory of FDE, Hardy–Poincaré inequalities (2) with $\gamma<0$ are the basic tools to investigate the large-time asymptotic of solutions [1, 2, 4, 6]. For example, the best constant in (2) is used in [3, 7] to show the fastest rate of convergence of solutions of fast diffusion equation and to bring some information about spectral properties of the

2010 *Mathematics Subject Classification*: Primary 26D10; Secondary 26D15, 35J60, 35R45.
Key words and phrases: Hardy inequalities, Hardy–Poincaré inequalities, *p*-harmonic problems, nonlinear eigenvalue problems
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc101-0-17

elliptic operator $L_{\alpha,d}u := -h_{1-\gamma}\operatorname{div}(h_{-\gamma}\nabla u)$, where $h_\alpha = (1+|x|^2)^\alpha$. We refer also to [4, 5, 16, 17] for the related results.

We are interested in (1) with $\gamma > 1$, and we take into account all $p \in (1,\infty)$, not only $p = 2$.

Our considerations are based on our recent result from [15], where we derived a one parameter family of Hardy type inequalities having the form

$$\int_\Omega |\xi|^p \,\mu_{1,\beta}(dx) \le \int_\Omega |\nabla\xi|^p \,\mu_{2,\beta}(dx),$$

where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is a compactly supported Lipschitz function, and Ω is an open subset of $\mathbb{R}^n$, not necessarily bounded. The involved measures $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$ depend on a certain parameter β and on u — a nonnegative weak solution to the partial differential inequality

$$-\Delta_p u \ge \Phi \quad \text{in} \quad \Omega, \tag{3}$$

with a locally integrable function Φ (see Theorem 2.3). The proof in [15] is inspired by the techniques from papers [10] and [14], dealing with the nonexistence of nontrivial nonnegative weak solutions to nonlinear problems in $\mathbb{R}^n$.

As a consequence, in [15] we retrieved the classical Hardy inequalities with optimal constants and obtained various weighted Hardy inequalities, among them those with radial measures.

In this paper we concentrate on (3) with $u_\alpha(x) = (1 + |x|^{\frac{p}{p-1}})^{-\alpha}$, $\alpha > 0$, and prove inequality (1) as well as optimality of the obtained constants for a range of parameters.

It appears that in some cases we improve the constants obtained by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [3], as well as those by Ghoussoub and Moradifam from [8]. In the case $p = 2$, $\gamma = n$, our constant is the same as in [3] and proven there to be optimal. Moreover, we show that our constants are also optimal for $p > 1$, when $\gamma \ge n + 1 - \frac{n}{p}$, but we do not know if they are optimal for a wider range of parameters, either in the case $p = 2$, or generally for $p > 1$. We finish this paper with a summary of the known values of constants, and their optimality, in different cases.

2. Preliminaries. In the sequel we assume that $p > 1$ and that Ω is an arbitrary open subset of $\mathbb{R}^n$. By p-harmonic problems we mean those which involve the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Definition 2.1 (Weighted Sobolev space). By $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where nonnegative measurable functions v_1, v_2 are given, we mean the completion of the set of functions $u \in C^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |u|^p \,v_1 dx < \infty$ and $\int_{\mathbb{R}^n} |\nabla u|^p \,v_2 dx < \infty$, under the norm

$$\|u\|_{W^{1,p}_{v_1,v_2}(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |u|^p \,v_1\, dx + \int_{\mathbb{R}^n} |\nabla u|^p \,v_2\, dx\Big)^{1/p}.$$

In [15] we derived Hardy–Poincaré inequalities from differential inequalities defined as follows.

Definition 2.2. Let Ω be an open subset of $\mathbb{R}^n$ and let Φ be a locally integrable function defined in Ω, such that for every nonnegative compactly supported $w \in W^{1,p}(\Omega)$

$$\int_\Omega \Phi w\, dx > -\infty. \tag{4}$$

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. We will say that

$$-\Delta_p u \geq \Phi,$$

if for every nonnegative compactly supported $w \in W^{1,p}(\Omega)$, we have

$$\langle -\Delta_p u, w\rangle := \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla w\rangle\, dx \geq \int_\Omega \Phi w\, dx.$$

In [15] we obtained the following result.

Theorem 2.3 ([15], Theorem 4.1). *Assume that $1 < p < \infty$ and that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_p u \geq \Phi$, in the sense of Definition 2.2, where Φ is locally integrable and satisfies the condition*

$$(\mathbf{\Phi}, \mathbf{p}) \qquad \sigma_0 := -\inf\big\{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma|\nabla u|^p \geq 0 \quad a.e.\ in\ \{u > 0\} \cap \Omega\big\} \in \mathbb{R},$$

where we set $\inf \emptyset = -\infty$. Assume further that β and σ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$.

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^p\, \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p\, \mu_2(dx), \tag{5}$$

where

$$\mu_1(dx) = \Big(\frac{\beta - \sigma}{p-1}\Big)^{p-1} \big[\Phi \cdot u + \sigma|\nabla u|^p\big] \cdot u^{-\beta-1}\chi_{\{u>0\}}\, dx, \tag{6}$$

$$\mu_2(dx) = u^{p-\beta-1}\chi_{\{|\nabla u| \neq 0\}}\, dx. \tag{7}$$

3. Main result. Hardy–Poincaré inequalities with optimal constants. In this part we show that application of Theorem 2.3 with a special function u, namely $u_\alpha(x) = (1 + |x|^{\frac{p}{p-1}})^{-\alpha}$ with $\alpha > 0$, leads to the following theorem.

Theorem 3.1. *Suppose $p > 1$ and $\gamma > 1$. Then, for every compactly supported function $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where $v_1(x) = \big(1 + |x|^{\frac{p}{p-1}}\big)^{(p-1)(\gamma-1)}$, $v_2(x) = \big(1 + |x|^{\frac{p}{p-1}}\big)^{(p-1)\gamma}$, we have*

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \big[(1 + |x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma-1}\, dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \big[(1 + |x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma}\, dx, \tag{8}$$

with $\bar{C}_{\gamma,n,p} = n\big(\frac{p(\gamma-1)}{p-1}\big)^{p-1}$. Moreover, for $\gamma > n + 1 - \frac{n}{p}$, the constant $\bar{C}_{\gamma,n,p}$ is optimal and it is achieved by the function $\bar{u}(x) = \big(1 + |x|^{\frac{p}{p-1}}\big)^{1-\gamma}$.

Proof. First we note that, by standard density argument, it suffices to prove (8) for every compactly supported Lipschitz function ξ. Indeed, let $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ and

$$\phi(x) = \begin{cases} 1, & |x| < 1, \\ -|x| + 2, & 1 \leq |x| \leq 2, \\ 0, & 2 < |x|. \end{cases} \qquad \phi_R(x) = \phi\Big(\frac{x}{R}\Big), \quad \xi_R(x) = \xi(x)\phi_R(x).$$

An easy verification shows that $\xi_R \to \xi$ in $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. A standard convolution argument shows that every compactly supported function $u \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ can be approximated in $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ by compactly supported Lipschitz functions.

Let us consider the function $u_\alpha(x) = (1+|x|^{\frac{p}{p-1}})^{-\alpha}$ with $\alpha > 0$. Now the proof follows by steps.

Step 1. We recognize that $u_\alpha \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and that it is a nonnegative solution to PDE

$$-\Delta_p(u_\alpha) = d(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}(1+\kappa|x|^{\frac{p}{p-1}}) =: \Phi \quad \text{a.e. in } \mathbb{R}^n, \tag{9}$$

where

$$d = d(n,\alpha,p) = \Big(\frac{\alpha p}{p-1}\Big)^{p-1} n \quad \text{and} \quad \kappa = \kappa(n,\alpha,p) = 1 - \frac{\alpha+1}{n}\, p. \tag{10}$$

Moreover, Φ satisfies (4). For the reader's convenience the computations are carried out in the Appendix.

Step 2. In our case condition $(\mathbf{\Phi},\mathbf{p})$ becomes

$$\sigma_0 := -\operatorname{ess\,inf}\Big(\frac{\Phi\cdot u_\alpha}{|\nabla u_\alpha|^p}\Big) = -\frac{p-1}{\alpha p}(n - p(\alpha+1)) \in \mathbb{R}. \tag{11}$$

Indeed, by the formulae (9) and (11), we have

$$\begin{aligned}
\sigma_0 &= -\inf \frac{\big(\frac{\alpha p}{p-1}\big)^{p-1}\big(1+|x|^{\frac{p}{p-1}}\big)^{-p(\alpha+1)}\big(n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}\big)}{\big(\frac{\alpha p}{p-1}\big)^{p}\big(1+|x|^{\frac{p}{p-1}}\big)^{-p(\alpha+1)}|x|^{\frac{p}{p-1}}}\\
&= -\inf \frac{n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}}{\big(\frac{\alpha p}{p-1}\big)|x|^{\frac{p}{p-1}}} = -\Big(\frac{p-1}{\alpha p}\Big)\Big[\inf \frac{n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}}{|x|^{\frac{p}{p-1}}}\Big]\\
&= -\frac{(p-1)(n-(\alpha+1)p)}{\alpha p}\,.
\end{aligned}$$

Step 3. For given $\alpha > -\gamma$, define $\beta = (p-1)(\frac{\gamma}{\alpha}+1)$. We apply Theorem 2.3.

For this we require that $\beta > 0$ and that $\sigma \in \mathbb{R}$ is such that $\beta > \sigma \geq \sigma_0$. This is equivalent to the condition $\gamma > \max\{-\alpha, 1-\frac{n}{p}\}$, which obviously holds for all $\gamma > 1$, $\alpha > 0$.

We are going to compute the measure given by (6). Let $b_1 = \big(\frac{\alpha p}{p-1}\big)^p \cdot \sigma$. We note that $\gamma = \alpha\big(\frac{\beta}{p-1} - 1\big)$ and $-p(\alpha+1)+\alpha(\beta+1) = (p-1)(\gamma-1)-1$ and recall that d and κ are given in (10). Applying these formulae to (6), we obtain

$$\begin{aligned}
\mu_1(dx) &= \Big(\frac{\beta-\sigma}{p-1}\Big)^{p-1}\big[\Phi\cdot u_\alpha + \sigma|\nabla u_\alpha|^p\big]u_\alpha^{-\beta-1}\,dx\\
&= \Big(\frac{\beta-\sigma}{p-1}\Big)^{p-1}\Bigg[\frac{d\big(1+\kappa|x|^{\frac{p}{p-1}}\big)}{\big(1+|x|^{\frac{p}{p-1}}\big)^{p(\alpha+1)}} + \frac{b_1|x|^{\frac{p}{p-1}}}{\big(1+|x|^{\frac{p}{p-1}}\big)^{p(\alpha+1)}}\Bigg]\cdot\big(1+|x|^{\frac{p}{p-1}}\big)^{\alpha(\beta+1)}\,dx\\
&= \Big(\frac{(\beta-\sigma)p\alpha}{(p-1)^2}\Big)^{p-1}\Big\{n + \Big[n-(\alpha+1)p+\frac{\sigma\alpha p}{p-1}\Big]|x|^{\frac{p}{p-1}}\Big\}\\
&\qquad\times\big(1+|x|^{\frac{p}{p-1}}\big)^{-1}\cdot\big[\big(1+|x|^{\frac{p}{p-1}}\big)^{p-1}\big]^{\gamma-1}\,dx,
\end{aligned} \tag{12}$$

while after substitution of $\beta = \frac{(p-1)(\alpha+\gamma)}{\alpha}$, we obtain from (7)

$$\mu_2(dx) = u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}\,dx = \Big[\big(1+|x|^{\frac{p}{p-1}}\big)^{-\alpha}\Big]^{p-\beta-1}\,dx = \Big[\big(1+|x|^{\frac{p}{p-1}}\big)^{p-1}\Big]^{\gamma}\,dx.$$

Step 4. We choose $\sigma := \frac{(p-1)(\alpha+1)}{\alpha}$ and realize that

$$\frac{(p-1)(\alpha+\gamma)}{\alpha} = \beta > \sigma > \sigma_0 = \frac{(p-1)(\alpha+1-n/p)}{\alpha},$$

because $\gamma > 1$. Then, in (12), the expression in curly brackets equals $n(1+|x|^{\frac{p}{p-1}})$. This leads to the inequality (8) with the constant as required.

Step 5. In this step we prove the optimality of the proposed constant under the assumption $\gamma > n+1-\frac{n}{p}$. It suffices to show that both sides of (8), for $u_\alpha := \bar{u}$ defined below, are equal and finite.

We prove first that the function $\bar{u}(x) = v(|x|) = (1+|x|^{\frac{p}{p-1}})^{1-\gamma}$ satisfies

$$-\operatorname{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u}) = \bar{C}_{\gamma,n,p}v_1\bar{u}^{p-1}. \tag{13}$$

For the reader's convenience the computations are carried out in the Appendix.

Now we concentrate on (8). Simple computations show that $\bar{u} \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. It suffices to prove equality in (8) for $\bar{u}$. Due to (13), we obtain

$$\begin{aligned}\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}|\bar{u}|^p\,\big(1+|x|^{\frac{p}{p-1}}\big)^{(p-1)(\gamma-1)}\,dx &= \bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}\bar{u}^p\,v_1\,dx\\ = -\int_{\mathbb{R}^n}\operatorname{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot\bar{u}\,dx &= -\lim_{R\to\infty}\int_{|x|<R}\operatorname{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot\bar{u}\,dx =: \mathcal{L}.\end{aligned}$$

We apply Gauss–Ostrogradski Theorem and observe that for an outer normal vector $n_x = \frac{x}{|x|}$ to $\partial B(R)$ we have $\langle\nabla\bar{u}, n_x\rangle = |\nabla\bar{u}|$. This implies

$$\mathcal{L} = \lim_{R\to\infty}\Big(\int_{|x|<R}v_2|\nabla\bar{u}|^p\,dx - \int_{|x|=R}v_2|\nabla\bar{u}|^{p-1}\cdot\bar{u}\,dS\Big) = \lim_{R\to\infty}(\mathcal{A}-\mathcal{B}),$$

where dS denotes the surface measure on the sphere $S^{n-1}(R)$. To deal with the limit we require $\gamma > n+1-\frac{n}{p}$. Let us observe, that $\lim_{R\to\infty}\mathcal{B} = 0$, because it is up to a constant equal to $\int_{|x|=R}\bar{u}(x)|x|\,dS$. Moreover, we notice that finiteness of the limit of $\mathcal{A}$ is ensured by

$$\frac{1}{\bar{C}_{\gamma,n,p}}\mathcal{A} \le \int_{\mathbb{R}^n}\big(1+|x|^{\frac{p}{p-1}}\big)^{-(\gamma-1)}\,dx \le \int_{\mathbb{R}^n}(1+|x|)^{-\frac{p(\gamma-1)}{p-1}}\,dx,$$

which is finite if the power of $(1+|x|)$ is smaller than $-n$, i.e. for $\gamma > n+1-\frac{n}{p}$.

This finishes the proof. ∎

Remark 3.2. Careful analysis of the quotient

$$\frac{b(R)}{a(R)} := \frac{\int_{\mathbb{R}^n}|\nabla u_R|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)\gamma}\,dx}{\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}|u_R|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)(\gamma-1)}\,dx}, \tag{14}$$

where $\bar{u}_R = \phi_R\bar{u}$, leads to optimality result also in the case of $\gamma = n+1-\frac{n}{p}$. We point out that when $\gamma = n+1-\frac{n}{p}$ the function $\bar{u}$ does not belong to $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. We will prove optimality in this case in another way in Corollary 4.3.

4. Discussion on constants

4.1. Comparison with the classical Hardy inequality. We start with showing that constants in Hardy–Poincaré inequalities are not smaller than in the classical Hardy inequalities. At first let us recall the classical results. We refer to [9, 11, 12] for more information on the best constants in various classical Hardy type inequalities.

THEOREM 4.1 (Classical Hardy inequalities). *Let $1 < p < \infty$.*

1. *Assume that $\gamma \neq p-1$ and ξ is an arbitrary Lipschitz function with compact support in $(0,\infty)$. Then*
$$\int_0^\infty \Big(\frac{|\xi|}{x}\Big)^p x^\gamma\,dx \le H_{\gamma,1,p}\int_0^\infty |\xi'|^p x^\gamma\,dx, \tag{15}$$
where the constant $H_{\gamma,1,p} = \big(\frac{p}{|p-1-\gamma|}\big)^p$ is optimal.
2. *Assume that $\gamma \neq p-n$ and ξ is an arbitrary Lipschitz function with compact support in $\mathbb{R}^n\setminus\{0\}$. Then*
$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi|^p|x|^{\gamma-p}\,dx \le H_{\gamma,n,p}\int_{\mathbb{R}^n\setminus\{0\}} |\nabla\xi|^p|x|^\gamma\,dx, \tag{16}$$
where the constant $H_{\gamma,n,p} = \big(\frac{p}{|p-n-\gamma|}\big)^p$ is optimal.

REMARK 4.2. The constant $HP_{\gamma,n,p} := 1/\bar{C}_{\gamma,n,p}$, where $\bar{C}_{\gamma,n,p}$ is the constant from Hardy–Poincaré inequality (8), is not smaller than the constant $H_{p\gamma,n,p}$ from Hardy inequality (16), namely
$$H_{p\gamma,n,p} \le HP_{\gamma,n,p}.$$

Proof. Let us consider (8) with function $\xi_t(y) := \xi(ty)$
$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n} |\xi(ty)|^p\big[(1+|y|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma-1}\,dy \le \int_{\mathbb{R}^n} t^p|\nabla\xi(ty)|^p\big[(1+|y|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma}\,dy,$$
and realize that it is equivalent to
$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n} |\xi(ty)|^p t^{-p(\gamma-1)}\big[(t^{\frac{p}{p-1}}+|ty|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma-1}\,dy \le \int_{\mathbb{R}^n} t^p|\nabla\xi(ty)|^p t^{-p\gamma}\big[(t^{\frac{p}{p-1}}+|ty|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma}\,dy.$$
We multiply both sides by $t^{p(\gamma-1)}$ and substitute $x = ty$, getting
$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n} |\xi(x)|^p\big[(t^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma-1}\,dx \le \int_{\mathbb{R}^n} |\nabla\xi(x)|^p\big[(t^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{p-1}\big]^{\gamma}\,dy.$$
It suffices to let $t\to 0$ and divide the inequality by $\bar{C}_{\gamma,n,p}$, to obtain
$$\int_{\mathbb{R}^n} |\xi(x)|^p|x|^{p(\gamma-1)}\,dy \le HP_{\gamma,n,p}\int_{\mathbb{R}^n} |\nabla\xi(x)|^p|x|^{p\gamma}\,dy. \tag{17}$$
We already know from Theorem 4.1 that the smallest possible constant is $H_{p\gamma,n,p}$. ∎

Applying this observation, we obtain the following result.

COROLLARY 4.3 (Optimal constant). *Suppose that $p > 1$, $n \geq 1$ and $\gamma = n(1-1/p)+1$. Then, for every nonnegative Lipschitz function ξ with compact support, inequality* (8) *holds with optimal constant $\bar{C}_{\gamma,n,p} = n^p$.*

Proof. We first notice that $HP_{\gamma,n,p} = HP_{n(1-1/p)+1,n,p} = \frac{1}{n}\left(\frac{p-1}{p(\gamma-1)}\right)^{p-1} = n^{-p} = \left(\frac{p\gamma}{|p\gamma-n-\gamma|}\right)^p = H_{p\gamma,n,p}$ (as $p\gamma \neq p-n$), and due to Remark 4.2 we recognize the optimality of this constant. ∎

4.2. Hardy–Poincaré inequalities with improved constants. In this section we concentrate on the classical case $p = 2$. We show that, for some values of parameters γ and n, our results improve the previously known constant in the Hardy–Poincaré inequality (2).

Links with results by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [2, 3]. In [2], the authors apply inequality (2) with $\gamma < 0$ to investigate convergence of solutions to fast diffusion equations. In [3], the following constants in (2) are established.

REMARK 4.4 ([3]). For every $v \in W^{1,2}_{v_1,v_2}(\mathbb{R}^n)$ where $v_1(x) = (1+|x|^2)^{\gamma-1}$, $v_2(x) = (1+|x|^2)^{\gamma}$, the inequality

$$\Lambda_{\gamma,n}\int_{\mathbb{R}^n}|v|^2(1+|x|^2)^{\gamma-1}\,dx \leq \int_{\mathbb{R}^n}|\nabla v|^2(1+|x|^2)^{\gamma}\,dx,$$

holds with $\Lambda_{\gamma,n}$ defined below.

1. For $n = 1$ and $\gamma < 0$ the optimal constant is

$$\Lambda_{\gamma,1} = \begin{cases} (\gamma-\frac{1}{2})^2 & \text{if } \gamma \in [-\frac{1}{2},0), \\ -2\gamma & \text{if } \gamma \in [-\infty,-\frac{1}{2}). \end{cases} \tag{18}$$

2. For $n = 2$ and $\gamma < 0$ the optimal constant is

$$\Lambda_{\gamma,2} = \begin{cases} \gamma^2 & \text{if } \gamma \in [-2,0), \\ -2\gamma & \text{if } \gamma \in [-\infty,-2). \end{cases} \tag{19}$$

3. For $n \geq 3$
   - and $\gamma < 0$ the optimal constant is

$$\Lambda_{\gamma,n} = \begin{cases} (n-2+2\gamma)^2/4 & \text{if } \gamma \in [-\frac{n+2}{2},0)\setminus\{-\frac{n-2}{2}\}, \\ -4\gamma-2n & \text{if } \gamma \in [-n,-\frac{n+2}{2}), \\ -2\gamma & \text{if } \gamma \in [-\infty,-n). \end{cases} \tag{20}$$

   - and $\gamma = n$ the optimal constant is $\Lambda_{n,n} = 2n(n-1)$,
   - and $\gamma \geq n$ the constant is $\Lambda_{\gamma,n} = n(n+\gamma-2)$,
   - and $n \geq \gamma > 0$ the constant is $\Lambda_{\gamma,n} = \gamma(n+\gamma-2)$.

REMARK 4.5. Here we compare our results with the above ones.

1. We preserve the optimal constant if $n \geq 3$ and $\gamma = n$.
2. We extend the above optimality result for $\gamma = n \geq 3$ also to the case $\gamma = n = 2$. Indeed, we recall that Corollary 4.3 applied to $p = 2$ gives the optimal constant

$\bar{C}_{(n+2)/2,n,2} = n^2$ when $n \geq 1$. In particular, we obtain $\Lambda_{2,2} = 2 \cdot 2(2-1) = \bar{C}_{(2+2)/2,2,2}$.

3. In the case $n \geq 3$, $\gamma > 2$, and $n \neq \gamma$, our constant $\bar{C}_{\gamma,n,2} = 2n(\gamma-1)$ is better than the constant in [3]:
   - if $\gamma > n$ then $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} = n(n+\gamma-2)$,
   - if $n > \gamma > 2$ then $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} = \gamma(n+\gamma-2)$.
4. In the case $n \geq 3$, $2 > \gamma > 1$ our constant becomes worse than $\Lambda_{\gamma,n}$.

Links with results by Ghoussoub and Moradifam [8]. In a recent paper [8] by Ghoussoub and Moradifam, some improvements to the results of [2] are obtained. In particular, some new estimates for constants from [2] are proven. We can further improve the constants from [8] for some range of parameters.

Among other results, one finds in [8] the following.

Theorem 4.6 ([8], Theorem 2.13, part II). *If $a, b, \alpha, \beta > 0$ and $n \geq 2$, then there exists a constant c such that for all $\xi \in C_0^\infty(\mathbb{R}^n)$*

$$c \int_{\mathbb{R}^n} (a + b|x|^\alpha)^{\beta - \frac{2}{\alpha}} \xi^2 \, dx \leq \int_{\mathbb{R}^n} (a + b|x|^\alpha)^\beta \, |\nabla \xi|^2 \, dx, \tag{21}$$

and moreover $\left(\frac{n-2}{2}\right)^2 =: c_1 \leq c \leq \left(\frac{n+\alpha\beta-2}{2}\right)^2$.

A very special case of the above theorem (when $a = b = 1$, $\alpha = 2$, and $\beta = \gamma$) covers also our case, therefore we present it below and discuss the related constants.

Corollary 4.7. *If $\gamma > 0$ and $n \geq 2$, then there exists a constant $\bar{c}_1 > 0$ such that for all $\xi \in C_0^\infty(\mathbb{R}^n)$*

$$\bar{c}_1 \int_{\mathbb{R}^n} |\xi|^2 (1 + |x|^2)^{\gamma - 1} \, dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^2 (1 + |x|^2)^\gamma \, dx, \tag{22}$$

and moreover $\left(\frac{n-2}{2}\right)^2 =: c_1 \leq \bar{c}_1 \leq \left(\frac{n+2\gamma-2}{2}\right)^2$.

Note that we have already pointed out in Remark 4.2 that $\bar{c}_1 \leq \left(\frac{n+2\gamma-2}{2}\right)^2$. Therefore, we may concentrate only on the lower bound.

Remark 4.8. Here we compare our results with the above one. The constant $\bar{C}_{\gamma,n,p}$ is the left-hand side constant derived in Theorem 3.1 for $\gamma, p > 1$, $n \geq 1$ and it is proven to be optimal for $\gamma \geq n + 1 - \frac{n}{p}$. Let c_1 be the constant from Corollary 4.7, where $\gamma > 0$, $p = 2$, $n \geq 2$. We may compare it only when $\gamma > 1$, $p = 2$, $n \geq 2$. We have

$$C_{\gamma,n,2} = 2n(\gamma - 1) > \left(\frac{n-2}{2}\right)^2 = c_1, \tag{23}$$

for every $\gamma > \max\left\{\frac{(n+2)^2}{8n}, 1\right\}$. This shows that for those γ's Theorem 3.1 gives the inequality (22) with the constant better than the one resulting from Corollary 4.7. Furthermore, we notice that (23) holds also for $\gamma \in \left(\frac{(n+2)^2}{8n}, 1 + \frac{n}{2}\right)$, when we do not have the optimality of $\bar{C}_{\gamma,n,2}$. When $\gamma = \frac{1}{2n}\left(\frac{n+2}{2}\right)^2$, we have $c_1 = \bar{C}_{\gamma,n,2}$, but for such γ we do not prove the optimality of $\bar{C}_{\gamma,n,2}$.

Comparison of the values of the constants $\bar{C}_{\gamma,n,2}$, $\Lambda_{\gamma,n}$, c_1 under common assumptions, in the case when $\bar{C}_{\gamma,n,2}$ is not proven to be optimal, is given in Remark 4.9.

4.3. Summary of results and open questions. We collect here all the known information about the constants in the Hardy–Poincaré inequality (1). We point out that we consider the left-hand side constant, and so the biggest possible one is optimal.

Let us recall that the constants c_1, $\Lambda_{\gamma,n}$ and $\bar{C}_{\gamma,n,p}$:

i) c_1 comes from [8], see Theorem 4.6 and Corollary 4.7,
ii) $\Lambda_{\gamma,n}$ comes from [3], see Remark 4.4,
iii) $\bar{C}_{\gamma,n,p}$ is derived in Theorem 3.1 for $p, \gamma > 1$, $n \geq 1$, and proven to be optimal
   – for $\gamma > \frac{n}{p}(p-1)+1$ in Theorem 3.1,
   – for $\gamma = \frac{n}{p}(p-1)+1$ in Corollary 4.3.

For $p = 2$, we have $\bar{C}_{\gamma,n,2} = 2n(\gamma-1)$, and moreover

n	γ	constant	optimality	see
$n \geq 1$	$\gamma > 1$	$\bar{C}_{\gamma,n,2}$	for $\gamma > \frac{n+2}{2}$, here	Thm 3.1
$n \geq 1$	$\gamma = \frac{n+2}{2}$	$\bar{C}_{\gamma,n,2}$	yes, here	Cor. 4.3
$n \geq 1$	$\gamma < 0$	$\Lambda_{\gamma,n}$	yes, [3]	Rem. 4.4
$n = 2$	$\gamma = 2$	$\bar{C}_{2,2,2}$	yes, here	Rem. 4.5
$n \geq 3$	$\gamma = n$	$\bar{C}_{n,n,2}$	yes, [3]	Rem. 4.4
$n \geq 3$	$\gamma > n$	$\bar{C}_{\gamma,n,2} \geq \Lambda_{\gamma,n} > c_1$	yes, here	Rem. 4.5
$n = 2$	$0 < \gamma < 1$	c_1	??	Cor. 4.7
$n \geq 3$	$\gamma \in (0, \min\{\gamma_c, 1\}]$	$c_1 \geq \Lambda_{\gamma,n}$	??	Cor. 4.7
$n \geq 3$	$\gamma_c \leq \gamma \leq 1$	$\Lambda_{\gamma,n} \geq c_1$	??	Cor. 4.7
$n \geq 2$	$1 < \gamma \leq \gamma_g$	$c_1 \geq \bar{C}_{\gamma,n,2}$	??	Cor. 4.7
$n \geq 2$	$\gamma > \gamma_g$	$\bar{C}_{\gamma,n,2} > c_1$	for $\gamma \geq \frac{n+2}{2}$, here	Rem. 4.8

where $\gamma_c = \frac{\sqrt{2}-1}{2}(n-2)$, $\gamma_g = \frac{(n+2)^2}{8n}$.

As we can see above, for sufficiently big values of γ ($\gamma \geq \frac{n+2}{2}$) our constant is optimal, thus $\bar{C}_{\gamma,n,2} \geq \max\{\Lambda_{\gamma,n}, c_1\}$. In the following remark we compare the values of the constants in the case when all three of them are defined (namely $p = 2$, $n \geq 3$, $\gamma > 1$) and when $\gamma < \frac{n+2}{2}$.

Remark 4.9. We compare all the mentioned constants under assumptions: $p = 2$, $n \geq 3$, and $1 < \gamma < \frac{n+2}{2}$. We note

i) $c_1 < \Lambda_{\gamma,n}$ if and only if $\gamma_c < \gamma$; $c_1 > \Lambda_{\gamma,n}$ if and only if $\gamma_c > \gamma$;
ii) $\bar{C}_{\gamma,n,2} < c_1$ if and only if $\gamma < \gamma_g$; $\bar{C}_{\gamma,n,2} > c_1$ if and only if $\gamma > \gamma_g$;
iii) $\bar{C}_{\gamma,n,2} < \Lambda_{\gamma,n}$ if and only if $\gamma < 2$; $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n}$ if and only if $\gamma > 2$.

Therefore for $p = 2$, $n \geq 3$, and $n > \gamma > 1$ we have $\gamma_c < \frac{n+2}{2}$, $1 < \gamma_g < \frac{n+2}{2}$, moreover

constants	γ	such γ exists for
$\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} > c_1$	$\gamma \in (\max\{2, \gamma_c\}, \frac{n+2}{2})$	$n \geq 3$
$\bar{C}_{\gamma,n,2} > c_1 > \Lambda_{\gamma,n}$	$\gamma \in (\gamma_g, \gamma_c)$	$n \geq 12$
$\Lambda_{\gamma,n} > \bar{C}_{\gamma,n,2} > c_1$	$\gamma \in (\gamma_g, 2)$	$n \in [3, 11]$
$\Lambda_{\gamma,n} > c_1 > \bar{C}_{\gamma,n,2}$	$\gamma \in (\max\{1, \gamma_c\}, \gamma_g)$	$n \in [3, 11]$
$c_1 > \Lambda_{\gamma,n} > \bar{C}_{\gamma,n,2}$	$\gamma \in (1, \min\{2, \gamma_c\})$	$n \geq 7$
$c_1 > \bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n}$	$\gamma \in (2, \gamma_g)$	$n \geq 12$

For $p>1$, $n\geq 1$, due to Theorem 3.1, we have $\bar{C}_{\gamma,n,p}=n\big(\frac{p(\gamma-1)}{p-1}\big)^{p-1}$, and

γ	constant	optimality
$\gamma\in(1,\frac{n}{p}(p-1)+1)$	$\bar{C}_{\gamma,n,p}=n\big(\frac{p(\gamma-1)}{p-1}\big)^{p-1}$	??
$\gamma=\frac{n}{p}(p-1)+1$	$\bar{C}_{\gamma,n,p}=n^p$	Corollary 4.3
$\gamma>\frac{n}{p}(p-1)+1$	$\bar{C}_{\gamma,n,p}=n\big(\frac{p(\gamma-1)}{p-1}\big)^{p-1}$	Theorem 3.1

Open questions

- We do not know the optimal constant in (22) for $\gamma<\frac{n}{2}+1$.
- We do not know the optimal constant in (8) for $\gamma<n+1-\frac{n}{p}$ and our methods do not give any estimates for the constant when $\gamma<1$.

5. Appendix

Proof of Step 1 of Proposition 3.1. We recall $u_\alpha(x)=\big(1+|x|^{\frac{p}{p-1}}\big)^{-\alpha}$ and compute first everything which is needed to find its p-Laplacian.

$$\begin{aligned}\nabla u_\alpha(x)&=-\alpha\big(1+|x|^{\frac{p}{p-1}}\big)^{-\alpha-1}\,\frac{p}{p-1}\,|x|^{\frac{p}{p-1}-1}\,\frac{x}{|x|}\\&=\frac{-\alpha p}{p-1}\big(1+|x|^{\frac{p}{p-1}}\big)^{-\alpha-1}|x|^{\frac{1}{p-1}}\,\frac{x}{|x|}\,,\\|\nabla u_\alpha(x)|&=\Big|\frac{\alpha p}{p-1}\Big|\big(1+|x|^{\frac{p}{p-1}}\big)^{-\alpha-1}|x|^{\frac{1}{p-1}},\\|\nabla u_\alpha(x)|^{p-2}&=\Big|\frac{\alpha p}{p-1}\Big|^{p-2}\big(1+|x|^{\frac{p}{p-1}}\big)^{-(\alpha+1)(p-2)}|x|^{\frac{p-2}{p-1}},\\|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x)&=-\frac{\alpha p}{p-1}\Big|\frac{\alpha p}{p-1}\Big|^{p-2}\big(1+|x|^{\frac{p}{p-1}}\big)^{-(\alpha+1)(p-1)}x=\kappa_1 x u_{(\alpha+1)(p-1)}(x),\end{aligned}$$

where $\kappa_1=\frac{-\alpha p}{p-1}|\frac{\alpha p}{p-1}|^{p-2}$.

Then (as $\alpha>0$) we have

$$\begin{aligned}&\Delta_p(u_\alpha(x))=\mathrm{div}\big(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x)\big)=\sum_i\frac{\partial(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x))}{\partial x_i}\\&=\kappa_1\sum_i\frac{\partial\big(u_{(\alpha+1)(p-1)}(x)x_i\big)}{\partial x_i}\\&=\kappa_1\Big(\sum_i\frac{\partial\big(u_{(\alpha+1)(p-1)}(x)\big)}{\partial x_i}\,x_i+u_{(\alpha+1)(p-1)}(x)\sum_i\frac{\partial x_i}{\partial x_i}\Big)\\&=\kappa_1\Big(\frac{-(\alpha+1)(p-1)p}{p-1}\big(1+|x|^{\frac{p}{p-1}}\big)^{-(\alpha+1)(p-1)-1}|x|^{\frac{1}{p-1}}\frac{\sum_i x_i^2}{|x|}+nu_{(\alpha+1)(p-1)}(x)\Big)\\&=\kappa_1\Big(-(\alpha+1)p\big(1+|x|^{\frac{p}{p-1}}\big)^{\alpha-\alpha p-p}|x|^{\frac{p}{p-1}}+nu_{(\alpha+1)(p-1)}(x)\Big)\\&=\Big(\frac{\alpha p}{p-1}\Big)^{p-1}\big(1+|x|^{\frac{p}{p-1}}\big)^{\alpha-\alpha p-p}\Big((\alpha+1)p|x|^{\frac{p}{p-1}}-n(1+|x|^{\frac{p}{p-1}})\Big).\end{aligned}$$

Therefore, our Φ has a form

$$\begin{aligned}\Phi &= -\operatorname{div}(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x))\\ &= \Big(\frac{\alpha p}{p-1}\Big)^{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}\big(n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}})\big). \quad \blacksquare\end{aligned}$$

Proof of (13) *in Step 5 of Theorem 3.1.* The proof follows from the technical lemmas below (Lemmas 5.1, 5.2 and 5.3). They show that, under assumption of Theorem 3.1, $\bar{u}$ satisfies an equation equivalent to equation (13). Therefore $\bar{u}$ satisfies (13) as well.

LEMMA 5.1. *Let $\bar{u}(x) = v(|x|) \in C^2(\mathbb{R}\setminus\{0\})$ be an arbitrary function, $\Phi_p(\lambda) = |\lambda|^{p-2}\lambda$, $v_2(r) = \big(1 + r^{\frac{p}{p-1}}\big)^{(p-1)\gamma}$ then*

i) $\nabla\bar{u}(x) = v'(|x|)\frac{x}{|x|}$,
ii) $\Phi_p'(\lambda) = (p-1)|\lambda|^{p-2}$,
iii) $(\Phi_p(\nabla\bar{u}(x))) = \Phi_p(v'(|x|))\cdot\frac{x}{|x|}$,
iv) $\operatorname{div}(\Phi_p(\nabla\bar{u})) = |v'(|x|)|^{p-2}\big((p-1)v''(|x|) + (n-1)\frac{v'(|x|)}{|x|}\big)$,
v) $\nabla v_2(|x|) = \gamma p(1+|x|^{\frac{p}{p-1}})^{\gamma(p-1)-1}|x|^{\frac{1}{p-1}}\frac{x}{|x|}$.

Proof. We reach the claims i)–iii) and v) by elementary calculations. Then applying i)–iii) we prove claim iv) as follows

$$\begin{aligned}(\Phi_p(\nabla\bar{u})) &= \operatorname{div}\big(\Phi_p(v'(|x|))\tfrac{x}{|x|}\big) = \nabla\big(\Phi_p(v'(|x|))\big)\cdot\tfrac{x}{|x|} + \Phi_p(v'(|x|))\operatorname{div}\big(\tfrac{x}{|x|}\big)\\ &= \Phi_p'(v'(|x|))\nabla v'(|x|)\cdot\tfrac{x}{|x|} + \Phi_p(v'(|x|))\tfrac{n-1}{|x|}\\ &= \tfrac{x}{|x|}\Phi_p'(v'(|x|))v''(|x|)\tfrac{x}{|x|} + \Phi_p(v'(|x|))\tfrac{n-1}{|x|}\\ &= \Phi_p'(v'(|x|))v''(|x|) + \Phi_p(v'(|x|))\tfrac{n-1}{|x|}\\ &= (p-1)|v'(|x|)|^{p-2}v''(|x|) + |v'(|x|)|^{p-2}v'(|x|)\tfrac{n-1}{|x|}. \quad \blacksquare\end{aligned}$$

LEMMA 5.2. *Equation* (13), *where $\bar{u}(x) = v(|x|) \in C^2(\mathbb{R}\setminus\{0\})$ is an arbitrary function, $v_1(r) = \big(1 + r^{\frac{p}{p-1}}\big)^{(p-1)(\gamma-1)}$, $v_2(r) = \big(1 + r^{\frac{p}{p-1}}\big)^{(p-1)\gamma}$, is equivalent to the equation*

$$\begin{aligned}-A := -\Big\{\Big(&(\gamma p + n - 1)|x|^{\frac{1}{p-1}} + \frac{n-1}{|x|}\Big)v'(|x|) + (p-1)\big(1+|x|^{\frac{p}{p-1}}\big)v''(|x|)\Big\}\\ &= \bar{C}_{\gamma,n,p}\big(1+|x|^{\frac{p}{p-1}}\big)^{-p+2}v^{p-1}(|x|)(v'(|x|))^{-(p-2)} =: B. \quad (24)\end{aligned}$$

Proof. We concentrate first on the left-hand side of (13):

$$\begin{aligned}-LHS &= \operatorname{div}(v_2\cdot\Phi_p(\nabla\bar{u})) = \nabla v_2\cdot\Phi_p(\nabla\bar{u}) + v_2\operatorname{div}(\Phi_p(\nabla\bar{u})) = I + II,\\ I &= \gamma p\big(1+|x|^{\frac{p}{p-1}}\big)^{\gamma(p-1)-1}|x|^{\frac{1}{p-1}}\frac{x}{|x|}\cdot\Big|v'(|x|)\frac{x}{|x|}\Big|^{p-2}v'(|x|)\frac{x}{|x|}\\ &= \gamma p\big(1+|x|^{\frac{p}{p-1}}\big)^{\gamma(p-1)-1}|x|^{\frac{1}{p-1}}|v'(|x|)|^{p-2}v'(|x|),\\ II &= \big(1+|x|^{\frac{p}{p-1}}\big)^{\gamma(p-1)}|v'(|x|)|^{p-2}\Big((p-1)v''(|x|) + v'(|x|)\frac{n-1}{|x|}\Big).\end{aligned}$$

Therefore,

$$-\,LHS = (1+|x|^{\frac{p}{p-1}})^{\gamma(p-1)-1}\,|v'(|x|)|^{p-2}$$
$$\times\Big((\gamma p+n-1)|x|^{\frac{1}{p-1}}v'(|x|)+\frac{n-1}{|x|}\,v'(|x|)+(p-1)(1+|x|^{\frac{p}{p-1}})v''(|x|)\Big),$$

while the right-hand side of (13) equals

$$RHS = \bar{C}_{\gamma,n,p}\big(1+|x|^{\frac{p}{p-1}}\big)^{(\gamma-1)(p-1)}v^{p-1}(|x|).$$

As $LHS = RHS$, by multiplying this equation by $(1+|x|^{\frac{p}{p-1}})^{-\gamma(p-1)+1}\,|v'(|x|)|^{-(p-2)}$, we obtain (24). ∎

Lemma 5.3. *If $\alpha = 1-\gamma < 0$, the function $v(x) = (1+|x|^{\frac{p}{p-1}})^{\alpha}$ satisfies* (24).

Proof. We will need the following computations, where we identify $v(x)$ with the one variable function $v(r)$

$$v' = \frac{\alpha p}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-1}r^{\frac{1}{p-1}},$$
$$v'' = \frac{\alpha p}{p-1}\Big(\frac{(\alpha-1)p}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-2}r^{\frac{2}{p-1}}+\frac{1}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-1}r^{-\frac{p-2}{p-1}}\Big)$$
$$= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2}\big((\alpha-1)pr^{\frac{2}{p-1}}+(1+r^{\frac{p}{p-1}})r^{-\frac{p-2}{p-1}}\big)$$
$$= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2}\big(((\alpha-1)p+1)r^{\frac{2}{p-1}}+r^{-\frac{p-2}{p-1}}\big)$$
$$= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2}r^{-\frac{p-2}{p-1}}\big(1+((\alpha-1)p+1)r^{\frac{p}{p-1}}\big),$$
$$\frac{v^{p-1}}{|v'|^{p-2}} = \frac{(1+r^{\frac{p}{p-1}})^{\alpha(p-1)}}{\big|\frac{\alpha p}{p-1}\big|^{p-2}(1+r^{\frac{p}{p-1}})^{(\alpha-1)(p-2)}r^{\frac{p-2}{p-1}}} = \Big|\frac{p-1}{\alpha p}\Big|^{p-2}r^{-\frac{p-2}{p-1}}\,\frac{(1+r^{\frac{p}{p-1}})^{\alpha(p-1)}}{(1+r^{\frac{p}{p-1}})^{(\alpha-1)(p-2)}}$$
$$= \Big|\frac{p-1}{\alpha p}\Big|^{p-2}r^{-\frac{p-2}{p-1}}(1+r^{\frac{p}{p-1}})^{\alpha+p-2}.$$

When we take into account the above results and substitute $\gamma = -\alpha+1$, we have on the first line of (24) the equality

$$-A = \Big((\gamma p+n-1)|x|^{\frac{1}{p-1}}+\frac{n-1}{|x|}\Big)v'(|x|)+(p-1)(1+|x|^{\frac{p}{p-1}})v''(|x|)$$
$$= \Big((\gamma p+n-1)|x|^{\frac{1}{p-1}}+\frac{n-1}{|x|}\Big)\frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{\frac{1}{p-1}}$$
$$+(p-1)(1+|x|^{\frac{p}{p-1}})\frac{(1-\gamma)p}{(p-1)^2}(1+|x|^{\frac{p}{p-1}})^{-\gamma-1}|x|^{-\frac{p-2}{p-1}}\big(1+(-\gamma p+1)|x|^{\frac{p}{p-1}}\big)$$
$$= \frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}\big((n-1)+(\gamma p+n-1)|x|^{\frac{p}{p-1}}\big)$$
$$+\frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}\big(1+(-\gamma p+1)|x|^{\frac{p}{p-1}}\big)$$
$$= n\,\frac{(1-\gamma)p}{p-1}\,(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}(1+|x|^{\frac{p}{p-1}}) = n\,\frac{(1-\gamma)p}{p-1}\,(1+|x|^{\frac{p}{p-1}})^{-\gamma+1}|x|^{-\frac{p-2}{p-1}}$$

and on the second line of (24)

$$\begin{aligned}
B &= \bar{C}_{\gamma,n,p}(1+|x|^{\frac{p}{p-1}})^{-p+2}\frac{v^{p-1}(|x|)}{|v'(|x|)|^{p-2}}\\
&= \bar{C}_{\gamma,n,p}(1+|x|^{\frac{p}{p-1}})^{-p+2}\Big(\frac{p-1}{(\gamma-1)p}\Big)^{p-2}|x|^{-\frac{p-2}{p-1}}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1+p-2}\\
&= n\Big(\frac{p(\gamma-1)}{p-1}\Big)^{p-1}\Big(\frac{p-1}{(\gamma-1)p}\Big)^{p-2}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1}|x|^{-\frac{p-2}{p-1}}\\
&= n(\gamma-1)\frac{p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1}|x|^{-\frac{p-2}{p-1}}.
\end{aligned}$$

We recognize that $-A = B$ for all $\gamma > 1$, $n \geq 1$, $p > 1$. ∎

Acknowledgments. The author would like to thank Agnieszka Kałamajska for multiple insightful comments. The author also thanks very much the referee for the comments, which significantly helped the author to improve the presentation of the article.

Research of the author was supported by NCN grant 2011/03/N/ST1/00111.

References

[1] M. Agueh, A. Blanchet, J. A. Carrillo, *Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime*, J. Evol. Equ. 10 (2010), 59–84.

[2] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J.-L. Vázquez, *Hardy–Poincaré inequalities and application to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris 344 (2007), 431–436.

[3] M. Bonforte, J. Dolbeault, G. Grillo, J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Natl. Acad. Sci. USA 107 (2010), 16459–16464.

[4] J. A. Carrillo, C. Lederman, P. A. Markowich, G. Toscani, *Poincaré inequalities for linearizations of very fast diffusion equations*, Nonlinearity 15 (2002), 565–580.

[5] J. Dolbeault, I. Gentil, A. Guillin, F.-Y. Wang, *L^q-functional inequalities and weighted porous media equations*, Potential Anal. 28 (2008), 35–59.

[6] J. Dolbeault, G. Toscani, *Fast diffusion equations. Matching large time asymptotics by relative entropy methods*, arXiv:1005.1994v2 [math.AP].

[7] M. Fila, J. L. Vázquez, M. Winkler, E. Yanagida, *Rate of convergence to Barenblatt profiles for the fast diffusion equation*, Arch. Ration. Mech. Anal. 204 (2012), 599–625.

[8] N. Ghoussoub, A. Moradifam, *Bessel pairs and optimal Hardy and Hardy–Rellich inequalities*, Math. Ann. 349 (2011), 1–57.

[9] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Univ. Press, Cambridge, 1952.

[10] A. Kałamajska, K. Pietruska-Pałuba, I. Skrzypczak, *Nonexistence results for differential inequalities involving A-Laplacian*, Adv. Differential Equations 17 (2012), 307–336.

[11] A. Kufner, *Weighted Sobolev Spaces*, Wiley, New York, 1985.

[12] A. Kufner, L. E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, River Edge, NJ, 2003.

[13] V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[14] E. Mitidieri, S. Pohozaev, *Nonexistence of positive solutions to quasilinear elliptic problems in* $\mathbb{R}^n$, Tr. Mat. Inst. Steklova 227 (1999), 192–222; English transl.: Proc. Steklov. Inst. Math. 227 (1999), 186–216.

[15] I. Skrzypczak, *Hardy-type inequalities derived from p-harmonic problems*, Nonlinear Anal. 93 (2013), 30–50.

[16] J. L. Vázquez, E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential*, J. Funct. Anal. 173 (2000), 103–153.

[17] F.-Y. Wang, *Orlicz–Poincaré inequalities*, Proc. Edinb. Math. Soc. (2) 51 (2008), 529–543.