A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter

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Based on joint work with Yousef Shakiba and Georg Zetzsche

Automata in the Wild 2025

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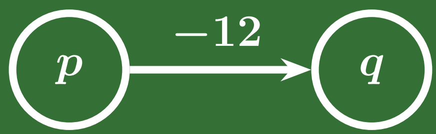
University of Warwick, UK

"Automata with One Counter" is a strange turn of phrase...

Integer Semantics

Transitions have updates in $\mathbb{Z}$.

The counter value is in $\mathbb{Z}$.



$$p(5) \rightarrow q(-7)$$

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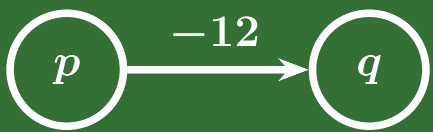

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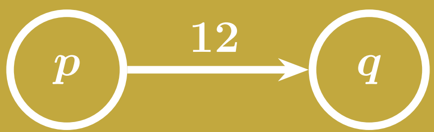

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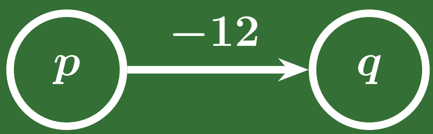

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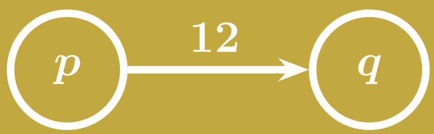

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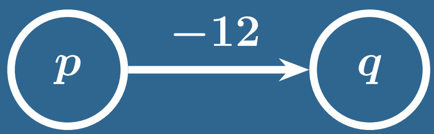

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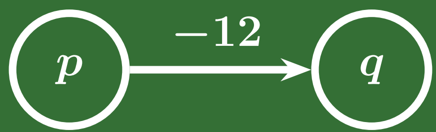

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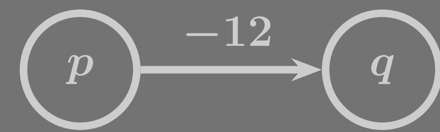

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Motivation

Reachability. From $p(x)$, can you reach $q(y)$?

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Theorem. Reachability (with integer semantics) is NP-hard.

Theorem. Coverability (with integer semantics) is in $\mathsf{AC}^1 \subseteq \mathsf{P}$.

Motivation

Reachability. From $\boldsymbol{p}(\boldsymbol{x})$, can you reach $\boldsymbol{q}(\boldsymbol{y})$?

Coverability. From $\boldsymbol{p}(\boldsymbol{x})$, can you reach $\boldsymbol{q}(\boldsymbol{y'})$ such that $\boldsymbol{y'} \geq \boldsymbol{y}$?

Theorem. Reachability (with integer semantics) is **HARD**

Theorem. Coverability (with integer semantics) is **EASY**

What makes reachability hard and coverability easy?

Generalising Reachability and Coverability

φ-Reachability

Fixed: A Presburger formula φ (with $p + 1$ variables).

Input: An integer one-counter automaton $\mathcal{A}$, an initial configuration $p(x)$, and a vector $\vec{t} \in \mathbb{Z}^p$.

Question: Does there exist a reachable configuation $q(y)$ such that $\varphi(\vec{t}, y)$ holds?

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Examples

Reachability: $\varphi(t, y) \Leftrightarrow y = t$.

Coverability: $\varphi(t, y) \Leftrightarrow y \geq t$.

Cover and avoid: $\varphi(t_1, t_2, y) \Leftrightarrow (y \geq t_1) \wedge (y \neq t_2)$.

Reach an interval: $\varphi(t, y) \Leftrightarrow y \in [t, 2t]$.

Reach an interval or a negative value: $\varphi(t, y) \Leftrightarrow y \in (-\infty, 0] \cup [t, 2t]$.

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Main Contribution

Theorem. Let φ be a Presburger formula with $p+1$ variables.

(1) If φ is **dense**, then φ-Reachability is in AC^1.

(2) Otherwise, φ-Reachability is NP-hard.

Density**: Let $S \subseteq \mathbb{Z}$ be a semilinear set... $\mathit{Density}(S) = \inf_{y \in S} \inf_{n \in \mathbb{N}} \frac{S \cap [y-n, y+n]}{2n+1}$

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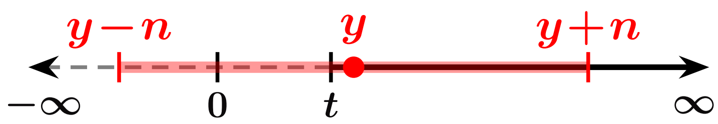

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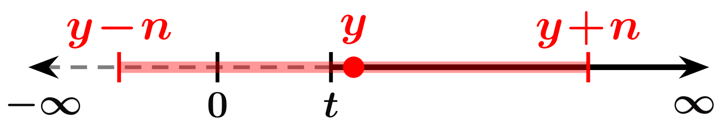

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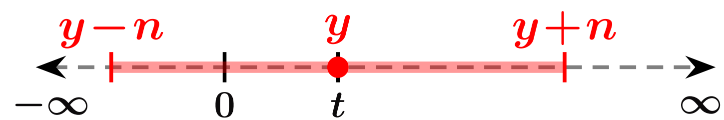

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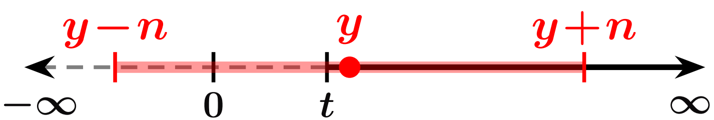

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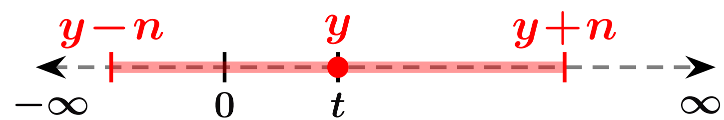

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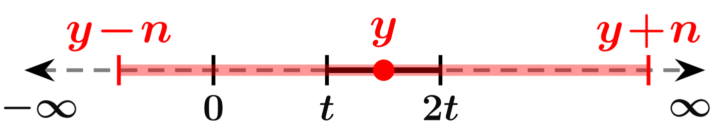

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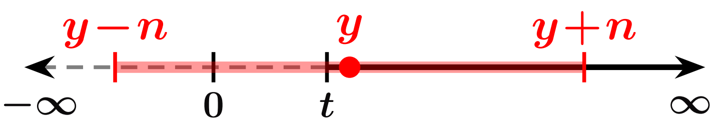

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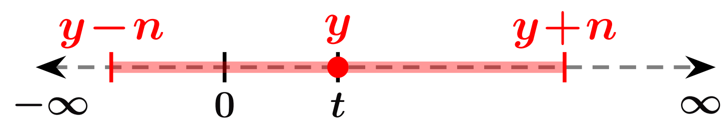

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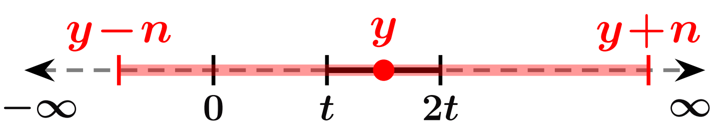

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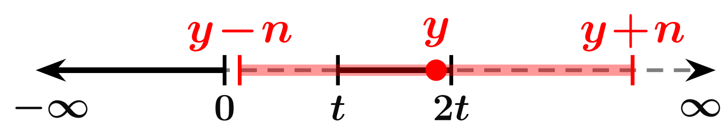

Density $= \frac{1}{4}$

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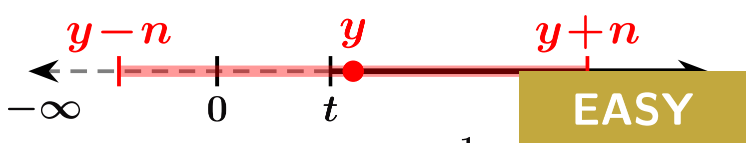

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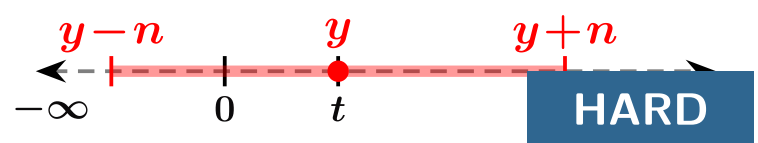

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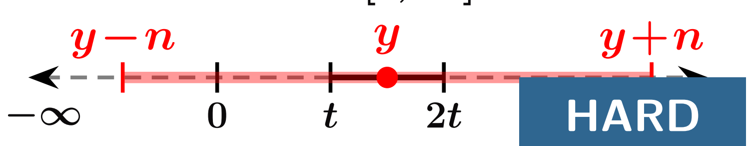

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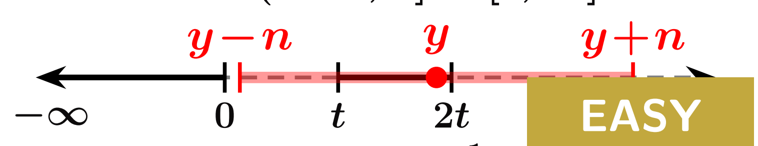

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φ is **dense** if $\quad$ $\inf_{\vec{t} \in \mathbb{Z}^p} \mathit{Density}(\{y : \varphi(\vec{t}, y) \text{ holds}\}) > 0$

** *simplified version ignoring modulo constraints.*

φ has **zero density**

$\Downarrow$

Reduction to subset sum

$\Downarrow$

φ-Reachability is NP-hard

φ has **positive density** $\implies$ φ-Reachability is in AC^1

Preliminaries:

Remove modulo constraints from φ.

Make the integer one-counter automaton acyclic.

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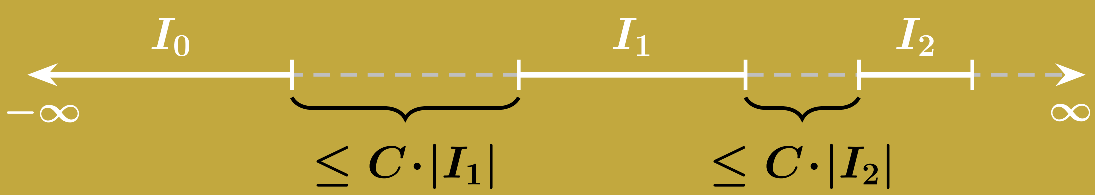

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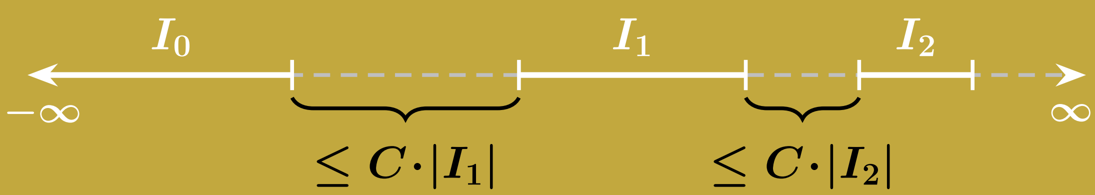

Design an AC^1 algorithm for reachability to *building blocks*.

View reachability in integer OCA as a semiring problem.

Use the Laurent polynomials with Boolean coefficients $\mathbb{B}[X, X^{-1}]$.

Relies on the technique of repeated squaring.

φ has **zero density**

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Reduction to subset sum

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Further Results

Let φ be a Presburger formula with $p+1$ variables.

Integer semantics: φ-$\textsc{Reachability}$ is ...
in AC^1 if φ is **dense**,
NP-hard otherwise.

Natural semantics: φ-$\textsc{Reachability}$ is ...
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$^+$*for a subtly modified definition of density*

VASS semantics: φ-$\textsc{Reachability}$ is ...
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Thank You!

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THE UNIVERSITY OF WARWICK

Presented by Henry Sinclair-Banks, University of Warsaw, Poland

Automata in the Wild 2025, University of Warwick, UK

Presentation made with `BeamerikZ`