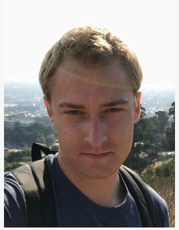
# The Tractability Border of Reachability in Simple Vector Addition Systems with States

**Henry Sinclair-Banks**

Based on work with Dmitry Chistikov, Wojciech Czerwiński, Filip Mazowiecki, Łukasz Orlikowski, and Karol Węgrzycki in FOCS'24.





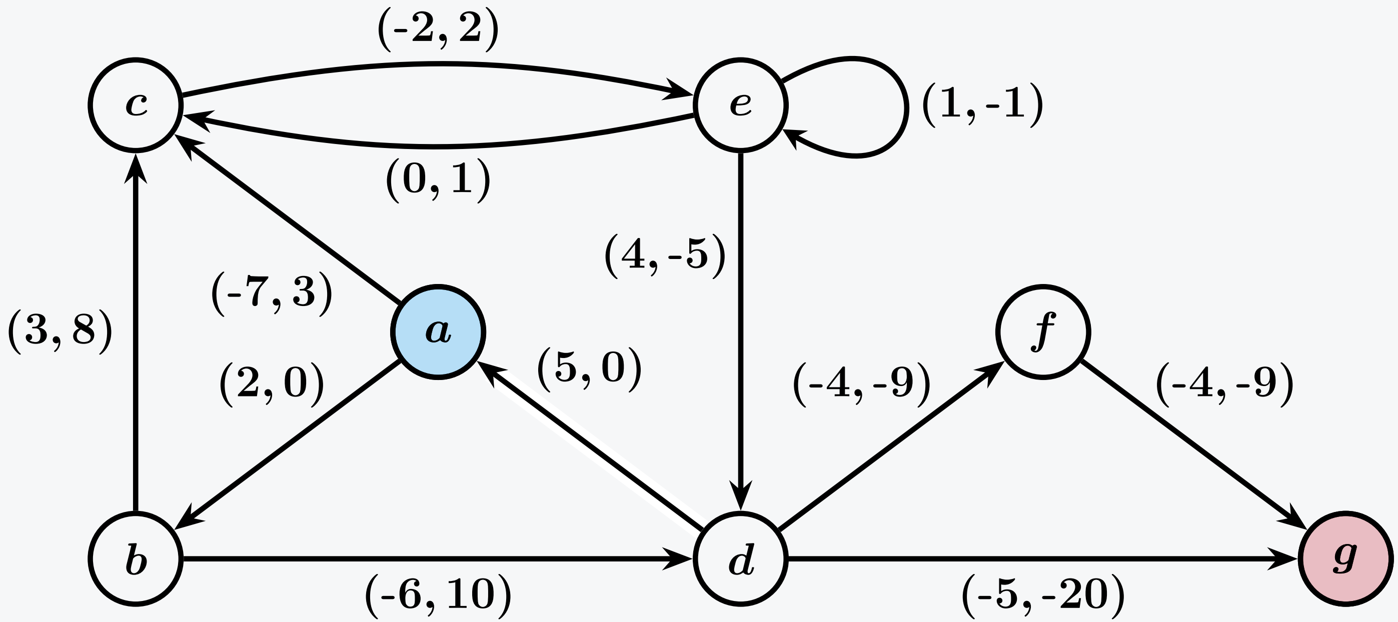



ISTA weekly seminar on verification

3rd December 2024

Institute of Science and Technology Austria, Vienna, Austria

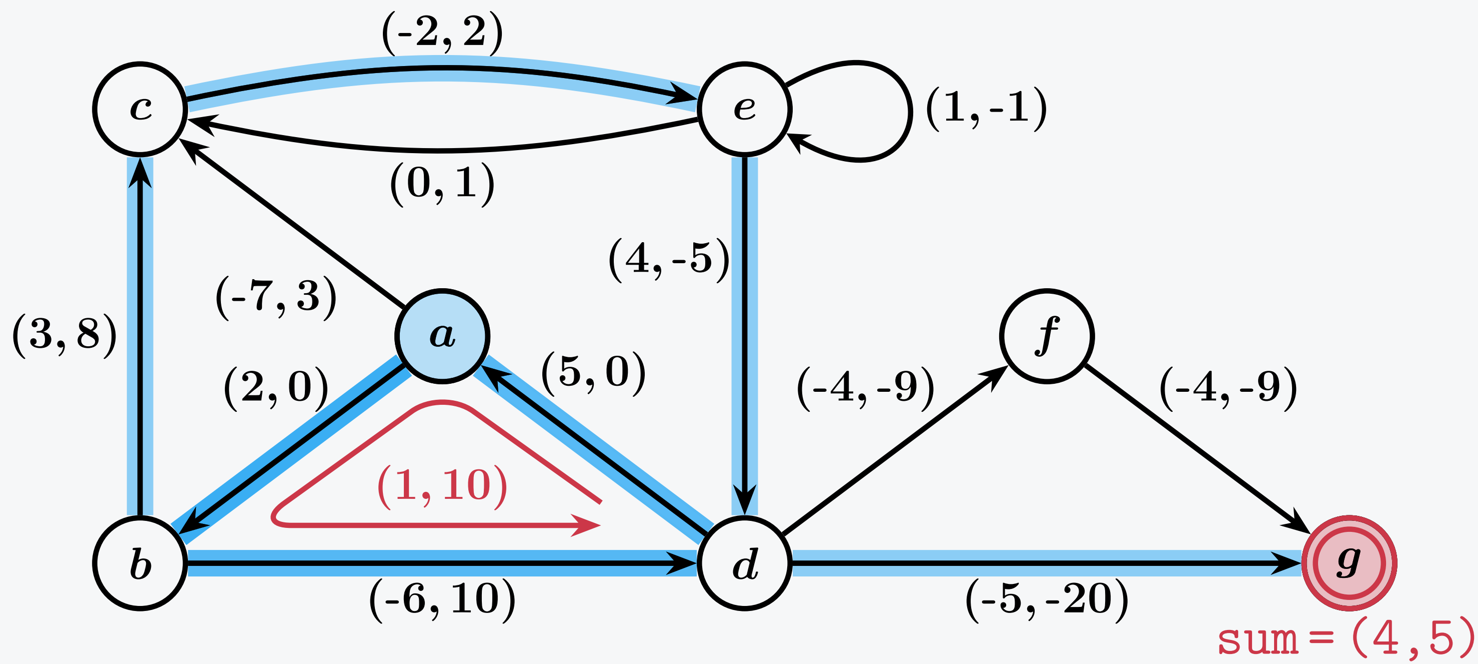
# Reachability in 2-Dimensional VASS




Does there exist a run from *a* with counter values (0,0) to *g* with counter values (4,5) ?

(the counters must remain nonnegative at all times)

# Reachability in 2-Dimensional VASS

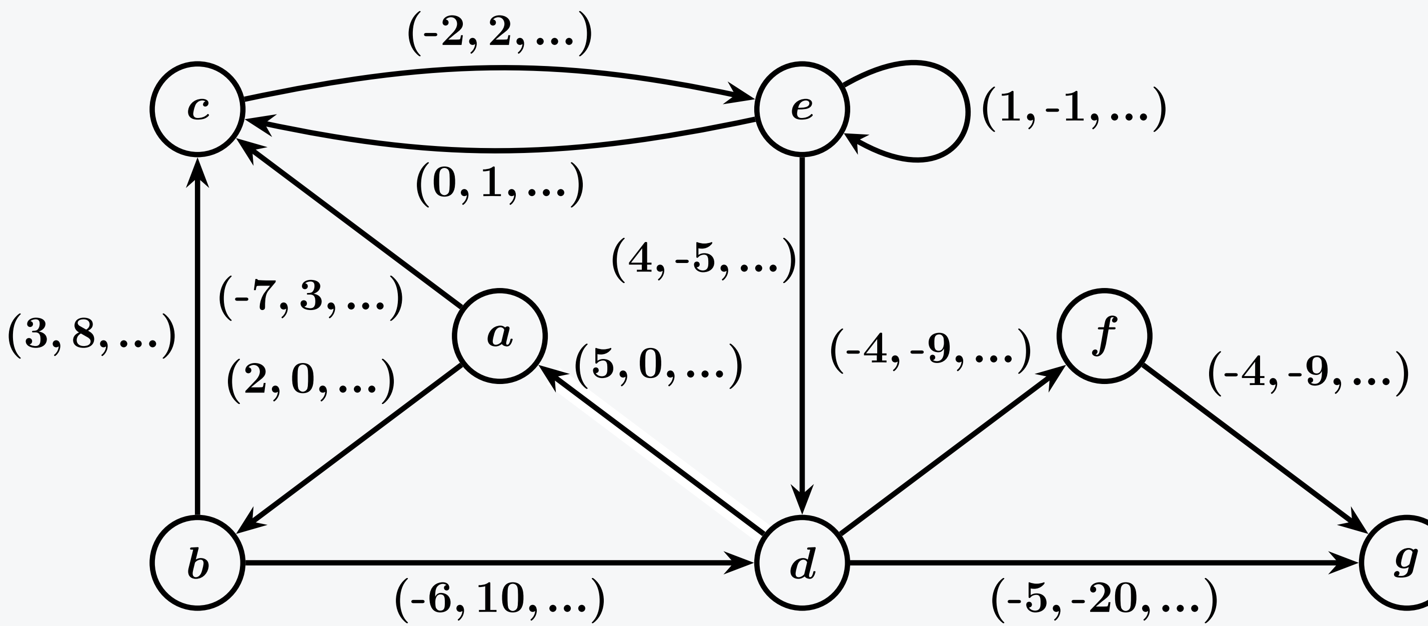



Does there exist a run from $a$ with counter values (0,0) to $g$ with counter values (4,5) ?

(the counters must remain nonnegative at all times)

**YES!**

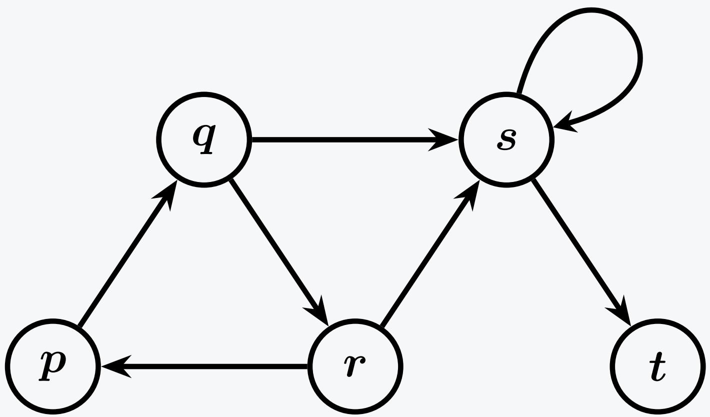
# Reachability in VASS



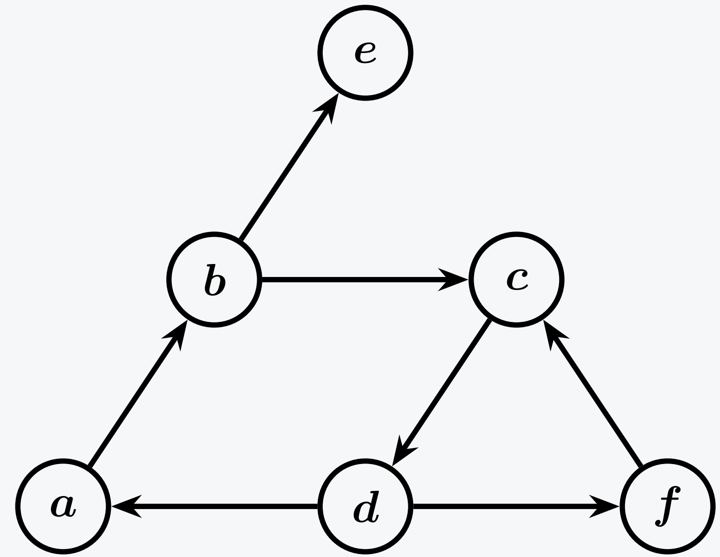

**Reachability problem:** does there exist a run from $p(\mathbf{u})$ to $q(\mathbf{v})$ ?

# “Simple” Vector Addition Systems with States

**Definition** (Flat)**.** For every state $q$, there is at most one simple cycle that contains $q$.




**Flat :)**



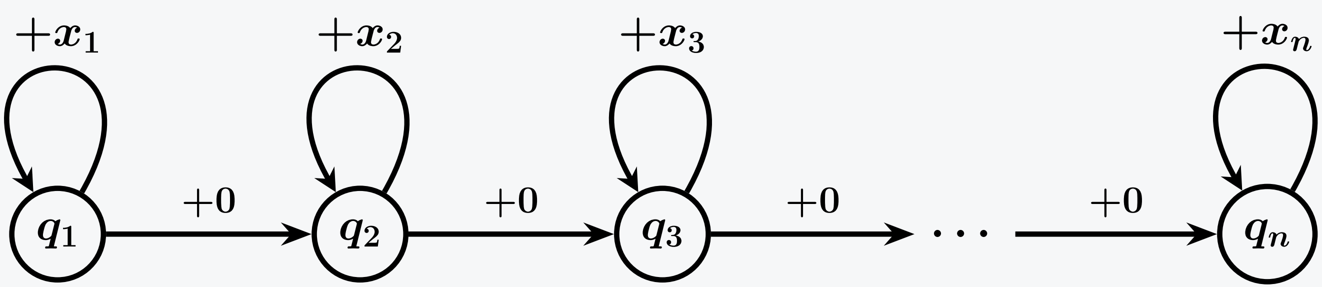

**Not flat :(**

# Reachability in Flat VASS

**Theorem.** Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

**Theorem.** Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

*Proof sketch.* Let $(\{x_1, \ldots, x_n\}, t)$ be an instance of subset sum (with multiplicities).




There exist $k_1, \ldots, k_n$ such that $\Sigma k_i \cdot x_i = t \implies$ there is a run from $q_1(0)$ to $q_n(t)$.

There is a run from $q_1(0)$ to $q_n(t) \implies$ there exist $k_1, \ldots, k_n$ such that $\Sigma k_i \cdot x_i = t$. □

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**Theorem.** Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

**Theorem.** Reachability in unary (flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73]

[Englert, Lazić, and Totzke '16]

**Theorem.** Reachability in unary flat $\boldsymbol{d}$-VASS is NP-hard for $\boldsymbol{d = 7}$.

[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

... for $\boldsymbol{d = 5}$. [Dubiak '20]

... for $\boldsymbol{d = 4}$. [Czerwiński and Orlikowski '22]

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What is the complexity of reachability in unary flat $\mathbf{3}$-VASS?

**Theorem.** Reachability in unary (flat) 1-VASS and 2-VASS is in NL.
[Blondin, Finkel, Göller, Haase, and McKenzie '15]
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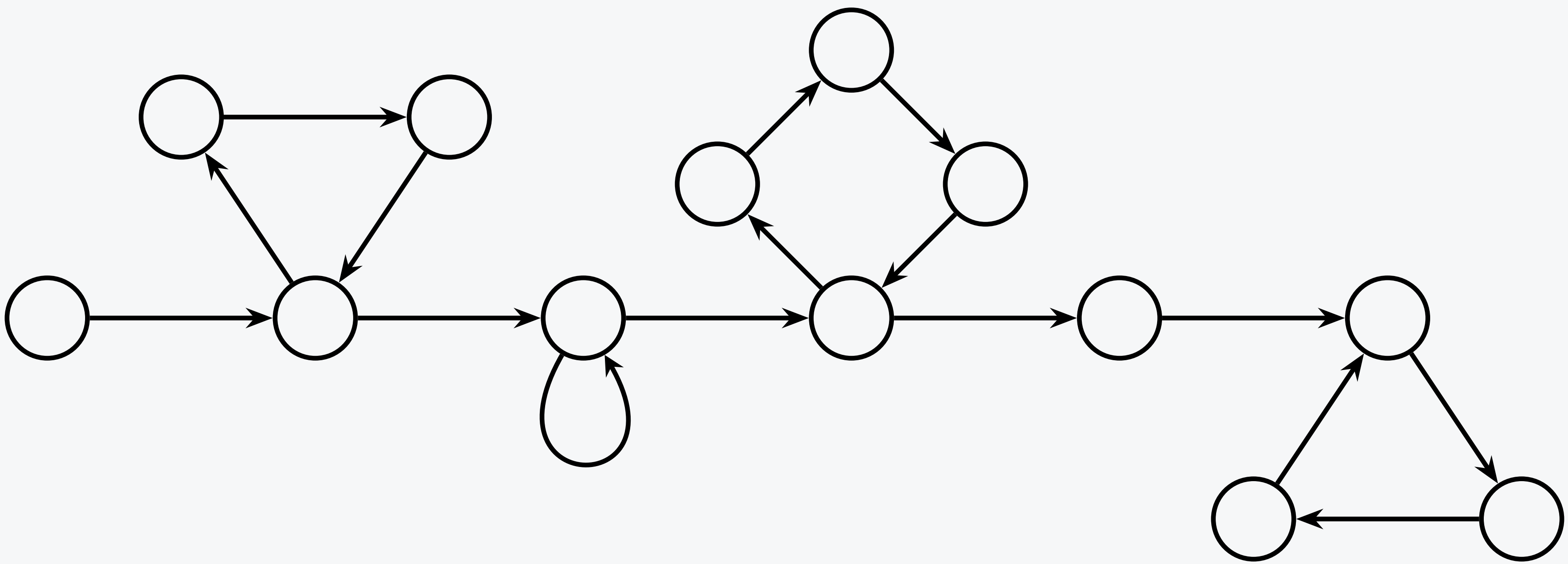
# ~~Flat VASS~~ Linear Path Schemes

**Definition** (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.



# ~~Flat VASS~~ Linear Path Schemes

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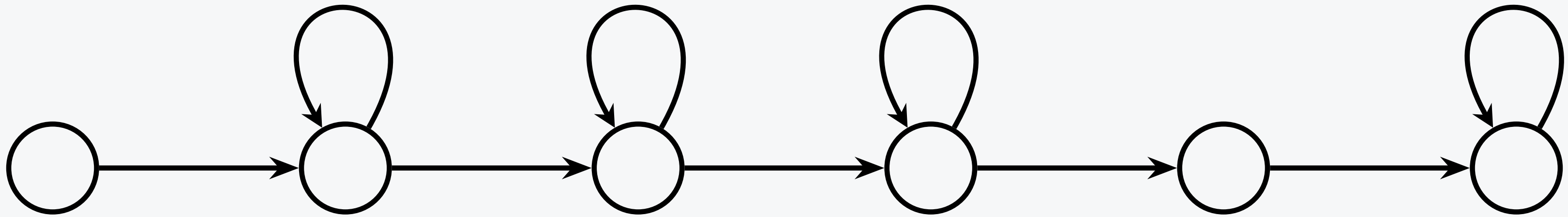
**Definition** (SLPS)**.** A *Simple* LPS has cycles of length one ("self-loops").



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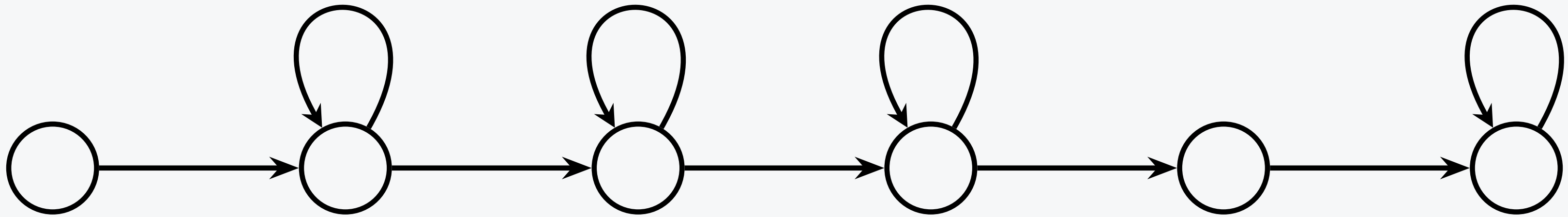

For $d \geq 3$, is reachability in unary $d$-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke ’16]

[Leroux ’21]

# Main Contribution

**Theorem 1.** Reachability in unary 3-SLPS is NP-complete.

*Proof approach.* Recall that reachability in (binary encoded) flat VASS is in NP.

For NP-hardness, reduce from 3-SAT.

1) Use “Chinese remainder encoding” for SAT.
2) Encode satisfiability as a conjunction of non-divisibility assertions.
3) Design a 2-SLPS with zero tests for asserting non-divisibility.
4) Concatenate these 2-SLPSs with zero tests so that reachability coincides with the conjunction of non-divisibility assertions.
5) Use an additional third counter to simulate the zero tests.

# Encoding SAT as a Conjunction of Non-Divisibility Assertions

Chinese remainder encoding for SAT with $k$ variables $x_1, \ldots, x_k$.

- Let $p_1, \ldots, p_k$ be the first $k$ primes.
- Let $n \in \mathbb{N}$ such that $n \equiv 0 \bmod p_i \iff x_i$ is false and $n \equiv 1 \bmod p_i \iff x_i$ is true.

First, enforce assignment validity.

- Want to verify that $n \equiv 0 \bmod p_i$ OR $n \equiv 1 \bmod p_i$ (for every $i$).
- Instead, check $p_i \nmid n - 2$ AND $p_i \nmid n - 3$ AND $\cdots$ AND $p_i \nmid n - (p_i - 1)$.

Second, enforce satisfiability.

- A clause $x_1 \vee \neg x_2 \vee x_3$ is satisfied if $n \equiv 1 \bmod 2$ OR $n \equiv 0 \bmod 3$ OR $n \equiv 1 \bmod 5$.
- This is only falsified when $n \equiv 10 \bmod 2 \cdot 3 \cdot 5$.
- Therefore, check $2 \cdot 3 \cdot 5 \nmid n - 10$.

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We were inspired by [Schöning '97]

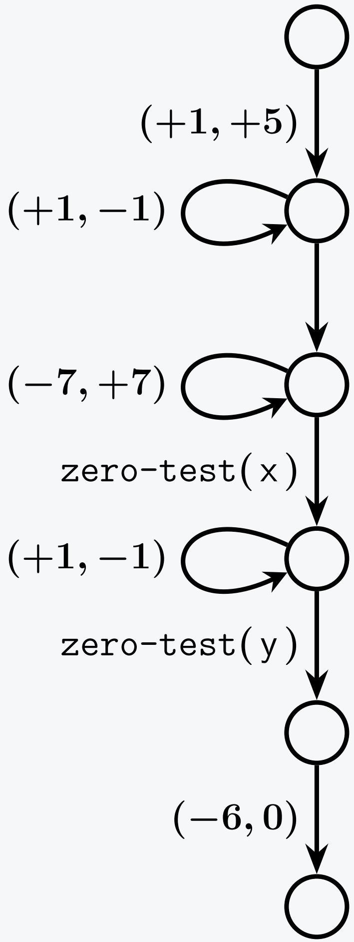
Idea first appears in [Stockmeyer and Meyer '73]

# Simple Linear Path Schemes Asserting Non-Divisibility

Suppose we want to assert $\mathbf{7} \nmid \boldsymbol{v}$.

Let's construct a 2-SLPS with zero tests that:

- starts with x $= \boldsymbol{v}$, y $= \mathbf{0}$,
- can only be passed if $\mathbf{7} \nmid \boldsymbol{v}$, and
- ends with x $= \boldsymbol{v}$, y $= \mathbf{0}$.



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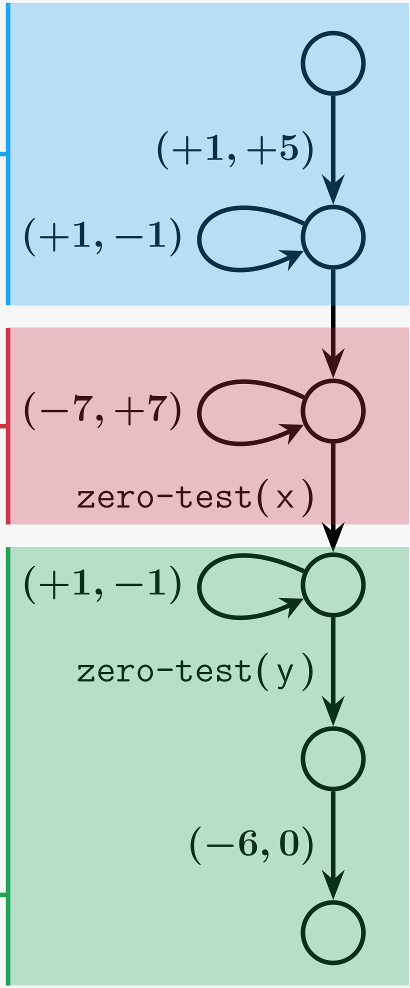
- ends with $\mathsf{x} = \boldsymbol{v}$, $\mathsf{y} = \mathbf{0}$.

**(i)** Choose $\boldsymbol{r} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$ ...

**(ii)** ... such that $\mathbf{7} \mid \boldsymbol{v} + \boldsymbol{r}$.

**(iii)** Restore $\mathsf{x} = \boldsymbol{v}$, $\mathsf{y} = \mathbf{0}$.



# Main Contribution

**Theorem.** Reachability in unary 3-SLPS is NP-complete.

*Proof approach.* Recall that reachability in (binary encoded) flat VASS is in NP.

For NP-hardness, reduce from 3-SAT.

✓ 1) Use "Chinese remainder encoding" for SAT.

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4) Concatenate these 2-SLPSs with zero tests so that reachability coincides with the conjunction of non-divisibility assertions. And add an $\mathrm{x} + 1$ loop for guessing an assignment $\mathrm{x} = \boldsymbol{v}$.

5) Use an additional third counter to simulate the zero tests.

# Simulating Zero Tests

**Lemma 2.2** (Controlling Counter Technique). *Let $\mathcal{Z}$ be a d-VASS with zero tests and let $s(\mathbf{x}), t(\mathbf{y})$ be two configurations. Suppose $\mathcal{Z}$ has the property that on any accepting run from $s(\mathbf{x})$ to $t(\mathbf{y})$, at most $m$ zero tests are performed on each counter. Then there exists a $(d+1)$-VASS $\mathcal{V}$ and two configurations $s'(\mathbf{0}), t'(\mathbf{y}')$ such that:*

*(1)* $s(\mathbf{x}) \xrightarrow{*}_{\mathcal{Z}} t(\mathbf{y})$ *if and only if* $s'(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}} t'(\mathbf{y}')$,

*(2)* $\mathcal{V}$ *can be constructed in* $\mathcal{O}((\mathit{size}(\mathcal{Z}) + \|x\|) \cdot (m+1)^d)$ *time, and*

*(3)* $\|\mathbf{y}'\| \leq \|\mathbf{y}\|$.

*Moreover, if $\mathcal{Z}$ is a flat VASS or a (simple) linear path scheme in which no zero-testing transition lies on a cycle, then $\mathcal{V}$ can be assumed to be a flat VASS or a (simple) linear path scheme, respectively.*

First developed in [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '19]

First formalised in [Czerwiński and Orlikowski '21]

Reformulated in [this paper]

**Takeway message:** A "small" number of zero tests can be simulated by an additional counter.

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# Main Results

**Theorem 1.** Reachability in unary 3-SLPS is NP-complete.

**Theorem 2.** Reachability in unary *ultraflat* 4-VASS is NP-complete.

**Theorem 3.** Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

**Theorem 4.** Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

# ~~Flat VASS~~ ~~Linear Path Schemes~~ Ultraflat VASS

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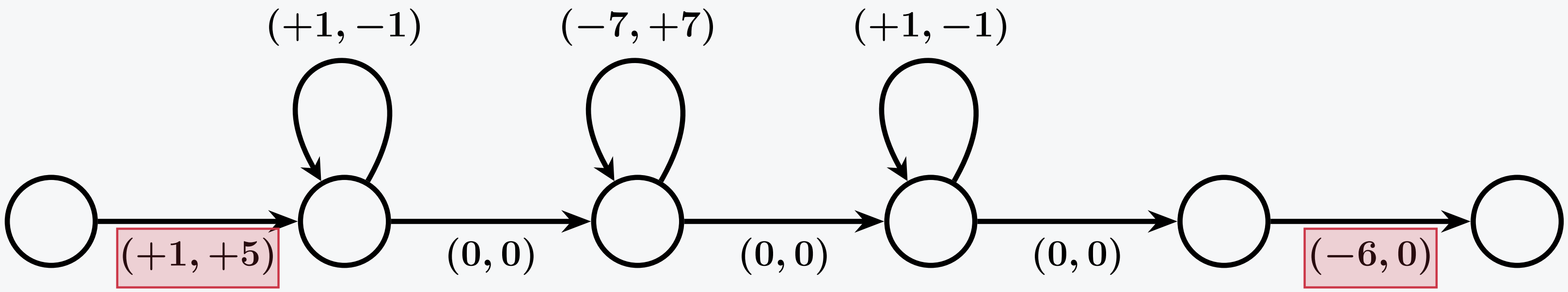
**Definition** (Ultraflat VASS)**.** An SLPS where the transitions between states have zero effect.




**Not ultraflat :(**

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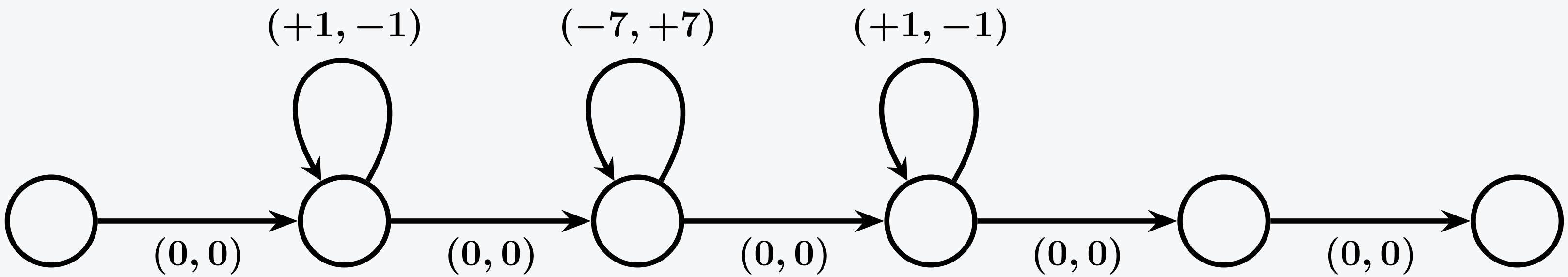


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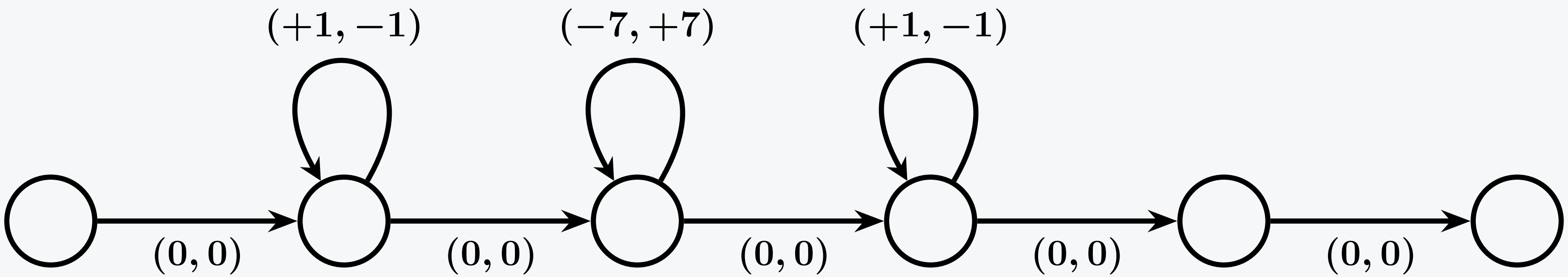



**Ultraflat :)**

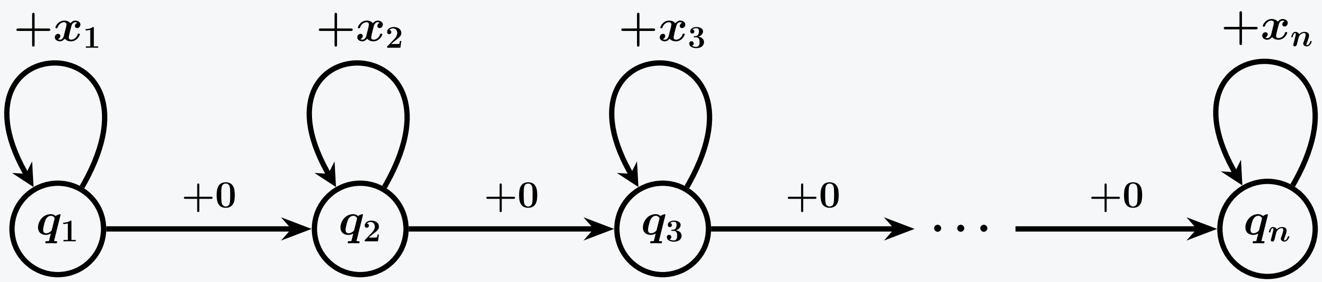
**Counter program notation:**

1. `LOOP:` x += 1, y −= 1
2. `LOOP:` x −= 7, y += 7
3. `LOOP:` x += 1, y −= 1

# Reachability in Ultraflat VASS

**Theorem.** Reachability in binary ultraflat 1-VASS is NP-hard. [Rosier and Yen '85] [Leroux '21]

*Proof idea.*




1. LOOP: z $+= x_1$
2. LOOP: z $+= x_2$
3. LOOP: z $+= x_3$
4. $\dots$

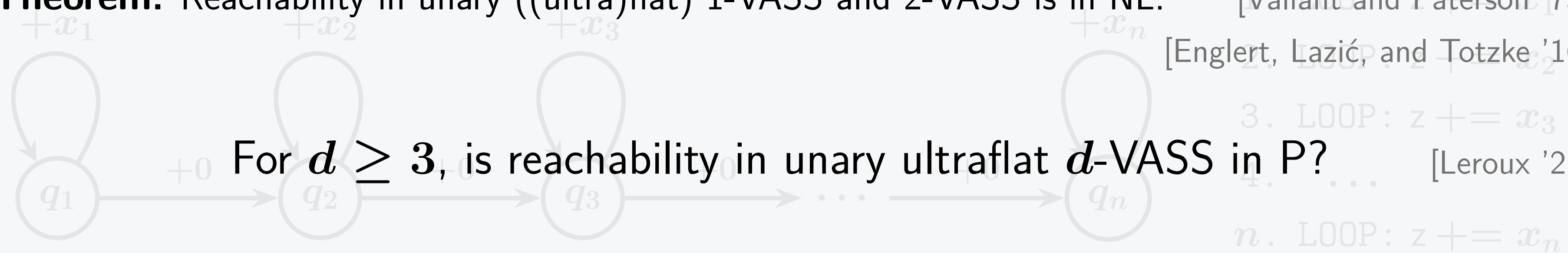
$n$. LOOP: z $+= x_n$

# Reachability in Ultraflat VASS

**Theorem.** Reachability in binary ultraflat 1-VASS is NP-hard. [Rosier and Yen '85] [Leroux '21]

*Proof idea.*

**Theorem.** Reachability in unary ((ultra)flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73] [Englert, Lazić, and Totzke '16]




For $d \geq 3$, is reachability in unary ultraflat $d$-VASS in P? [Leroux '21]

**Theorem 2.** Reachability in unary ultraflat 4-VASS is NP-complete.

*Proof ingredients.*

3-SAT reduction.

Chinese remainder encoding for SAT.

Conjunction of non-divisibility assertions.

*Ultraflat 3-VASS with zero tests for asserting non-divisibility.*

Concatenate non-divisibility asserting ultraflat VASS.

Simulate zero tests with a controlling counter.

# Recap of Main Results

**Theorem 1.** Reachability in unary 3-SLPS is NP-complete.

**Theorem 2.** Reachability in unary ultraflat 4-VASS is NP-complete.

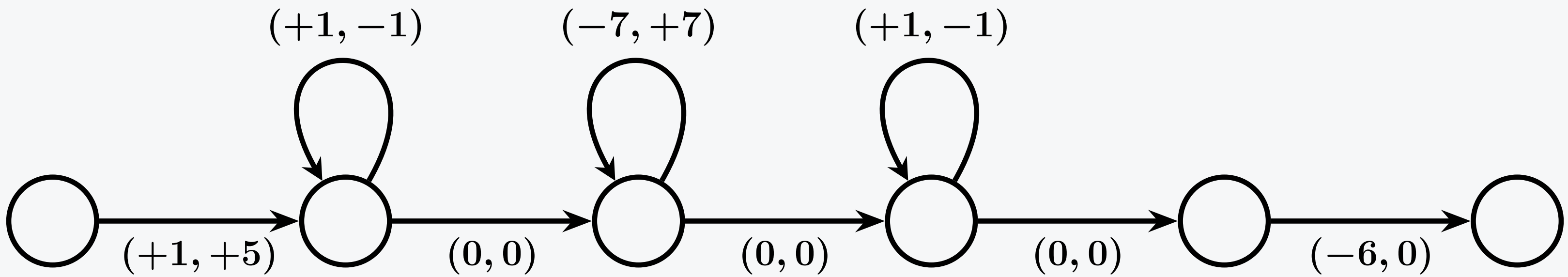
**Open problem.** Is reachability in unary ultraflat 3-VASS NP-complete?

**Theorem 3.** Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

**Theorem 4.** Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

# Unitary Simple Linear Path Schemes

**Definition** (Unitary SLPS)**.** An SLPS where the counter updates are restricted to $\{-1, 0, +1\}$.



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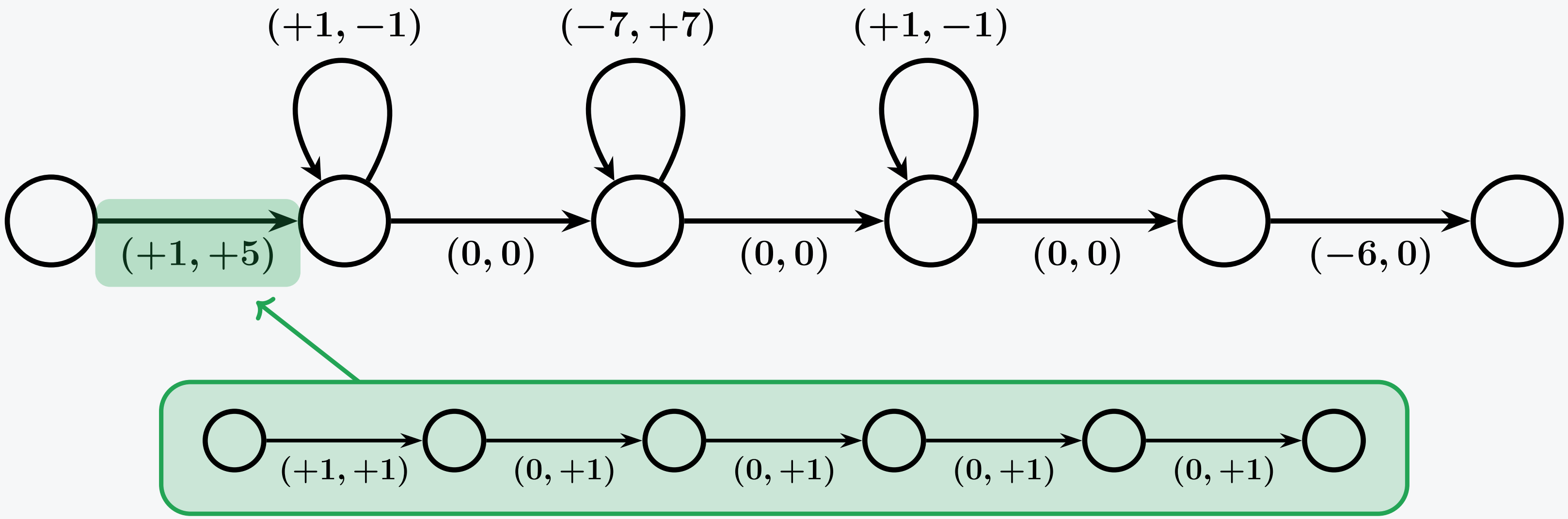

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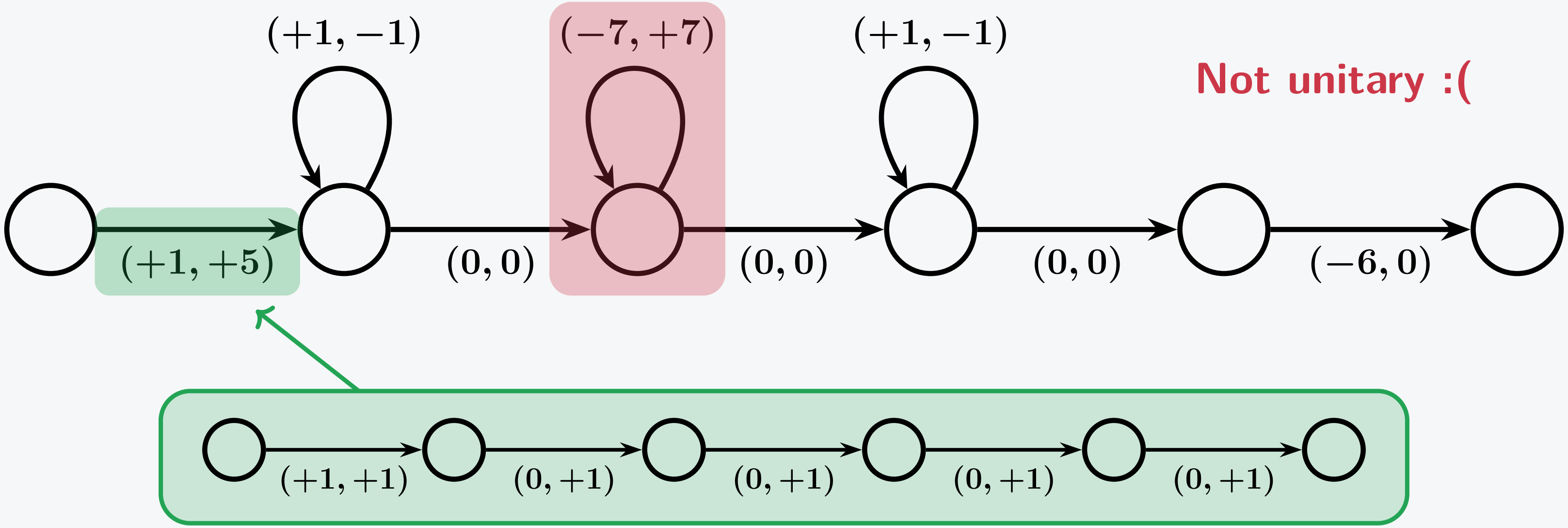

# Reachability in Unitary SLPS

**Theorem 3.** Reachability in unitary inverse-Ackermann-dimensional SLPS is NP-complete.

UnitaryInverseAckermannDimensionalSimpleLinearPathSchemeReachability

`Input:` a natural number $k$ encoded in unary,

a unitary $\mathcal{O}(\alpha(k))$-SLPS $\mathcal{V}$ of size $\textit{\textbf{poly}}(k)$,

an initial configuration $p(u)$ encoded in unary, and

a target configuration $q(v)$ encoded in unary.

`Question:` is there a run from $p(u)$ to $q(v)$ in $\mathcal{V}$ ?

Notation: $\alpha : \mathbb{N} \to \mathbb{N}$ is the inverse Ackermann function.

# Reachability in Unitary SLPS

**Theorem 3.** Reachability in unitary inverse-Ackermann-dimensional SLPS is NP-complete.

UNITARYINVERSEACKERMANNDIMENSIONALSIMPLELINEARPATHSCHEMEREACHABILITY

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`Question:` is there a run from $p(u)$ to $q(v)$ in $\mathcal{V}$?

*Proof ingredients.* 3-SAT reduction. Chinese remainder encoding for SAT.

Conjunction of non-divisibility assertions.

*Unitary 5-SLPS with zero tests for asserting non-divisibility.*

Concatenate non-divisibility asserting unitary SLPSs. *Simulate zero tests with a different technique.*

Notation: $\alpha : \mathbb{N} \to \mathbb{N}$ is the inverse Ackermann function.

# The Tractability Border of Reachability in Simple Vector Addition Systems with States

**Theorem 1.** Reachability in unary 3-SLPS is NP-complete.

**Theorem 2.** Reachability in unary ultraflat 4-VASS is NP-complete.

**Open problem.** Is reachability in unary ultraflat 3-VASS NP-complete?

**Theorem 3.** Reachability in unitary inverse-Ackermann-dimensional SLPS is NP-complete.

**Open problem.** Does there exist $d \in \mathbb{N}$ such that reachability in unitary $d$-SLPS is NP-complete?

**Theorem 4.** Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

## Thank You!

*Presented by Henry Sinclair-Banks, University of Warsaw, Poland*

*Weekly verification seminar, ISTA, Austria*

# Ultraflat 3-VASS with Zero Tests for Asserting Non-Divisibility

```
LOOP: x += 1, y += 5, z −= 1
LOOP: x += 2, y += 6, z −= 1
LOOP: x += 3, y += 7, z −= 1
LOOP: x += 4, y += 8, z −= 1
zero-test(z)
LOOP: x −= 5, z += 5
zero-test(x)
LOOP: x += 1, z −= 1
zero-test(z)
LOOP: x −= 1, y −= 5, z += 1
LOOP: x −= 2, y −= 6, z += 1
LOOP: x −= 3, y −= 7, z += 1
LOOP: x −= 4, y −= 8, z += 1
zero-test(y)
```

Suppose we want to assert $5 \nmid v$.

This ultraflat 3-VASS with zero tests:

- starts with $\mathsf{x} = v$, $\mathsf{y} = 0$, $\mathsf{z} = 1$,
- can only be passed if $5 \nmid v$, and
- ends with $\mathsf{x} = v$, $\mathsf{y} = 0$, $\mathsf{z} = 1$.

**(i)** Choose $r \in \{1, 2, 3, 4\}$ ...

**(ii)** ... such that $5 \mid v + r$.

**(iii)** Restore $\mathsf{x} = v$, $\mathsf{y} = 0$.