

Coverability in VASS Revisited: Improving Rackoff's Bound to Obtain Conditional Optimality

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About joint work with Marvin Künnemann, Filip Mazowiecki, Lia Schütze, and Karol Węgrzycki in ICALP'23.

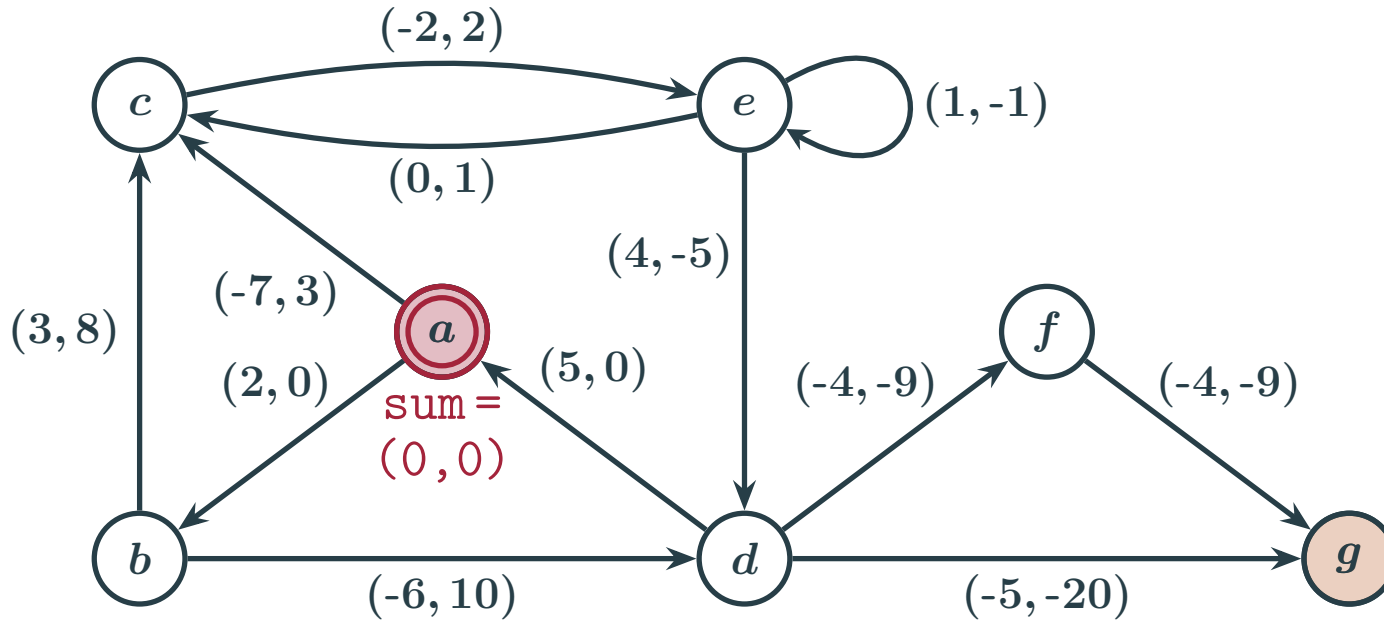


Verification Seminar

8th January 2024

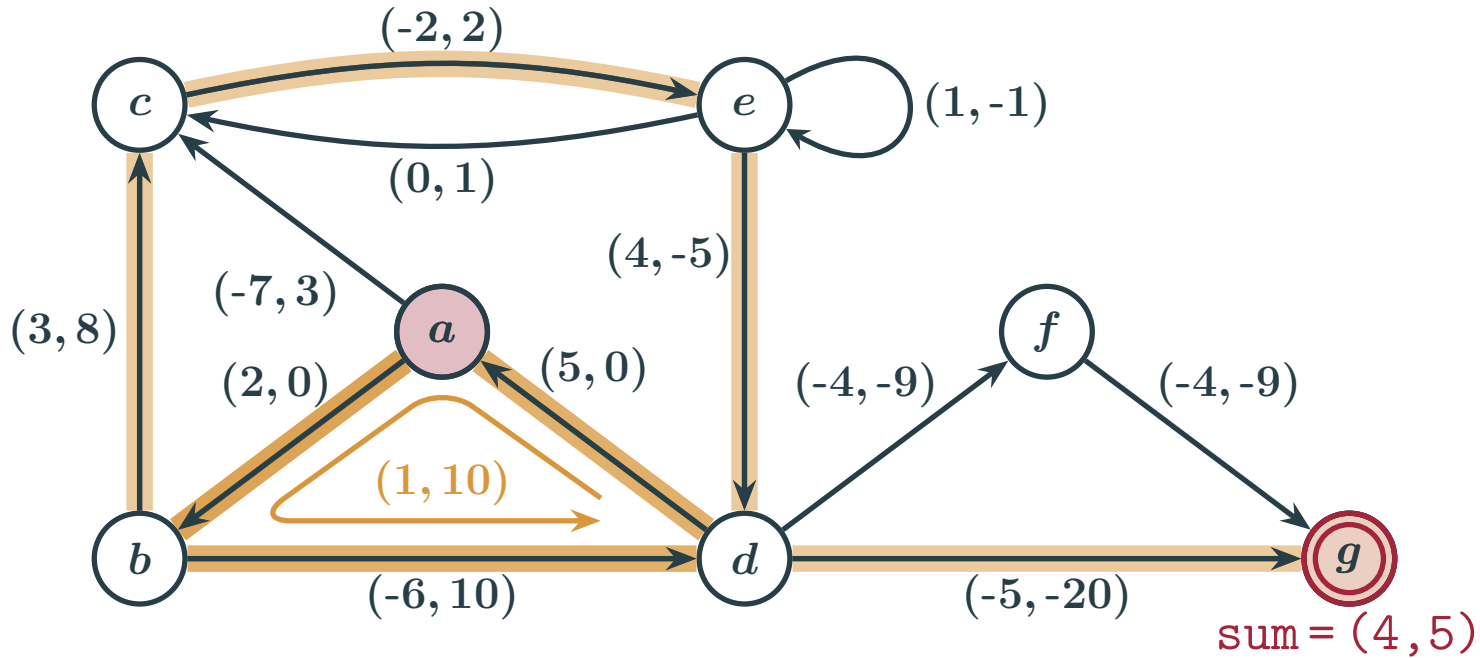
IRIF, Paris, France

Instance of Coverability in 2-Dimensional VASS



Question: from a can you reach g via a path that is *never negative on any component* ?

Instance of Coverability in 2-Dimensional VASS



Question: from a can you reach g via a path that is *never negative on any component* ? YES!

Motivation

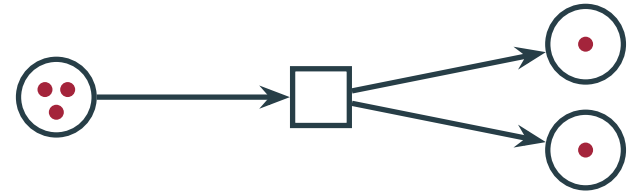
Resource Management



Road cost: $(-1L \text{ fuel}, +2kWh \text{ battery})$

Model of Concurrency

VASS are equivalent to Petri nets



Testing Safety

Positive instance of coverability



Some action sequence reaches a 'bad' state



System is unsafe!

Related Problems

Unboundedness

Reachability

Word problems for (commutative) semi-groups

Overview of this Presentation

1. The history and complexity of coverability.
2. Our improvement over Rackoff's upper bound.
Main concepts: introducing 'thin configurations' and using Rackoff's bounding technique.
3. Obtaining an optimal space algorithm and a conditionally optimal time algorithm.
4. Our Exponential Time Hypothesis conditional lower bound.
Main concepts: reducing clique detection to coverability and simulating bounded counter machines.

History and Complexity

d is the dimension: number of components.

n is the size: number of states plus the absolute values of all updates.
(unary encoding)

History and Complexity



Richard Lipton

Theorem: Coverability in VASS is EXPSPACE-hard.

[Lipton '76]



Charles Rackoff

Theorem: Coverability in VASS is in EXPSPACE.

[Rackoff '78]

d is the dimension: number of components.

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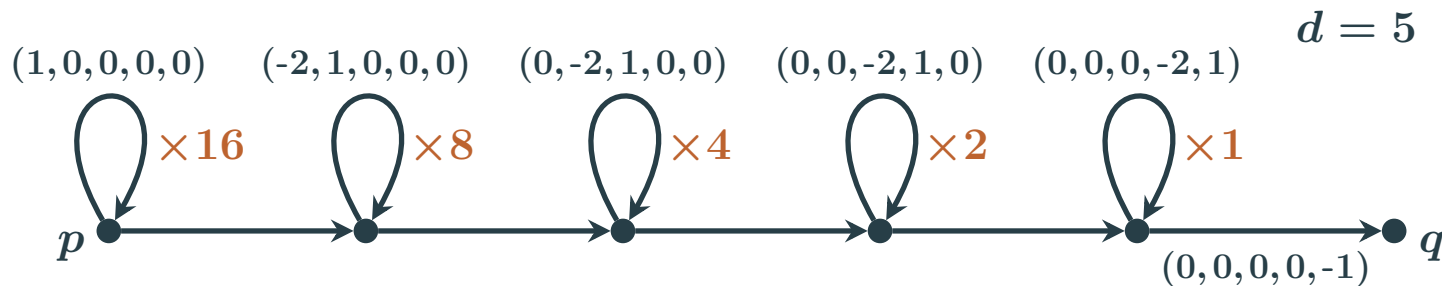
History and Complexity



Richard Lip

Example of Long Coverability Runs

[Lipton '76]



Any coverability run from p to q has length $2^{\Omega(d)}$.

d is the dimension: number of components.

n is the size: number of states plus the absolute values of all updates (unary encoding).

History and Complexity



Richard Lipton

Theorem: Coverability in VASS requires $2^{\Omega(d)} \cdot \log(n)$ space.

[Lipton '76]

Idea: find instances only admitting $n^{2^{\Omega(d)}}$ length runs. “*Lipton’s construction*”



Charles Rackoff

Theorem: Coverability in VASS can be decided in $2^{\mathcal{O}(d \log d)} \cdot \log(n)$ space.

[Rackoff '78]

Idea: argue that there are always $n^{2^{\mathcal{O}(d \log d)}}$ length runs. “*Rackoff’s bound*”



Ernst Mayr



Albert Meyer

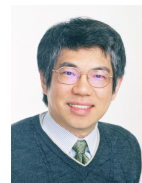
Open Problem

Improve these bounds.

[Mayr and Meyer '82]



Louis Rosier



Hsu-Chun Yen

Refined via a multiparameter analysis.

[Rosier and Yen '85]

d is the dimension: number of components.

n is the size: number of states plus the absolute values of all updates (unary encoding).

Vector Addition Systems with(out) States

d -VASS

$$(Q, T)$$

Q is a finite set of states.

$T \subseteq Q \times \mathbb{Z}^d \times Q$ are the transitions.

Configurations are in $Q \times \mathbb{N}^d$.

d -VAS

$$(V)$$

$V \subseteq \mathbb{Z}^d$ is just a set of vectors.

Configurations are in $\mathbb{N}^d$.



John
Hopcroft



Jean-Jacques
Pansiot

Lemma: A d -VASS can be *simulated* by a $(d + 3)$ -VAS. [Hopcroft and Pansiot '79]

Idea: maintain invariants containing information about the number of states and the current state on three dedicated additional counters.

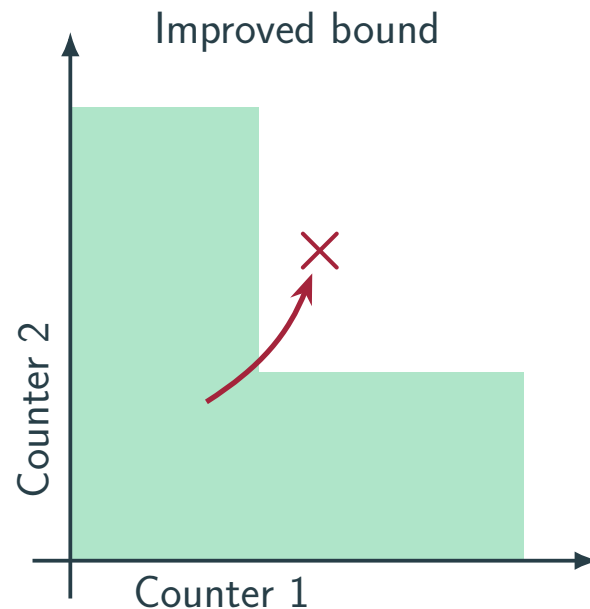
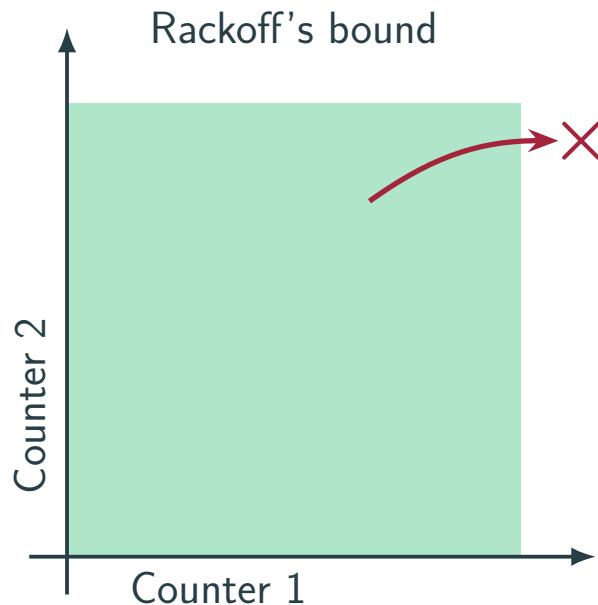
Takeaway: we will work with VAS because we do not fix the dimension.

Improving Rackoff's Upper Bound

Theorem: Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Idea: Carefully use Rackoff's bounding technique with sharper counter value bounds.



Improving Rackoff's Upper Bound

Theor

Idea:

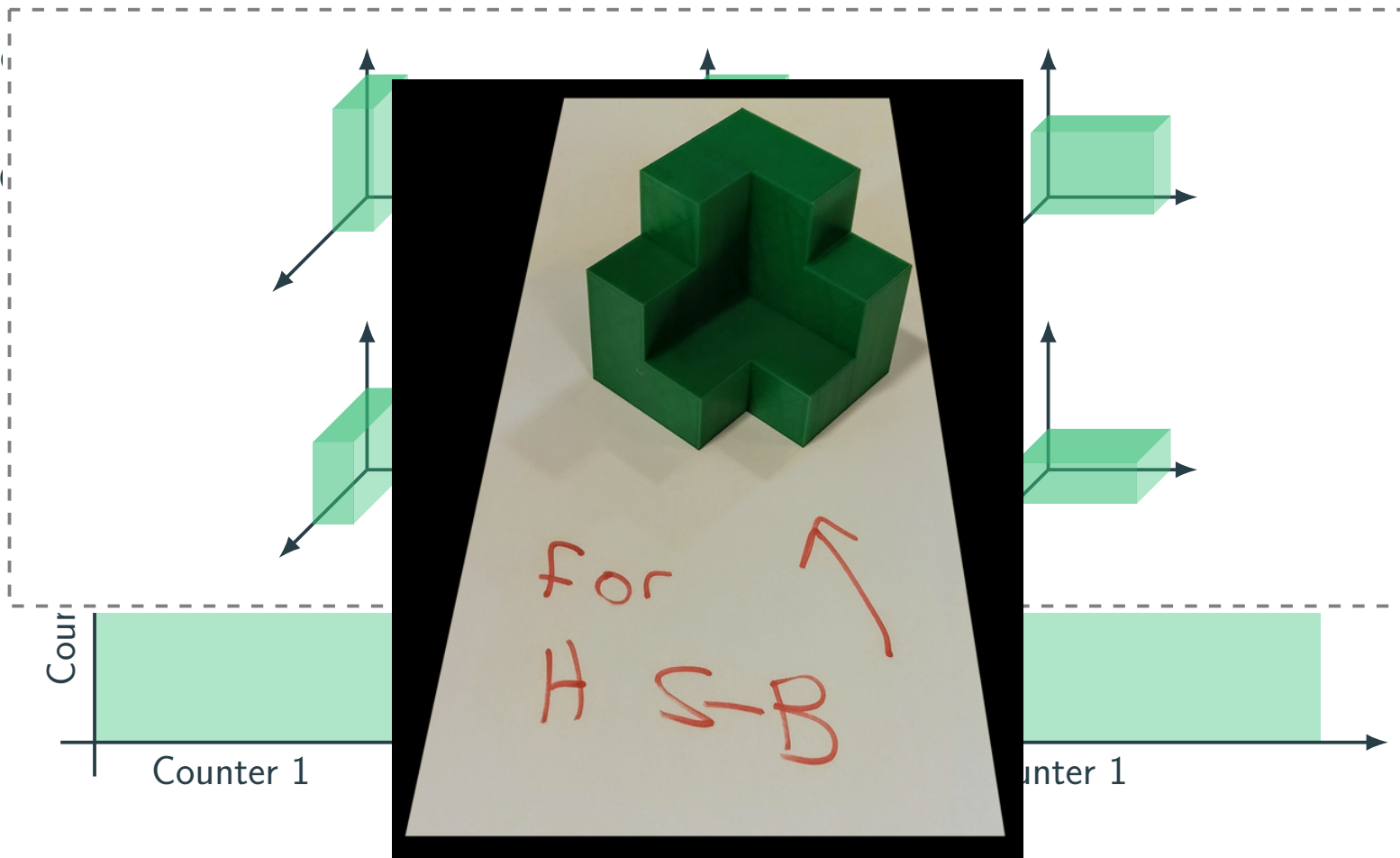
[grzycki '23]



Improving Rackoff's Upper Bound

Theor

Idea:



Thin Configurations

Definition: A configuration $\vec{v} \in \mathbb{N}^d$ is *thin* if, after sorting the components, $\vec{v}[1] < M_1$, $\vec{v}[2] < M_2$, ..., $\vec{v}[d] < M_d$.

Importantly, to get an improvement over Rackoff's bound:

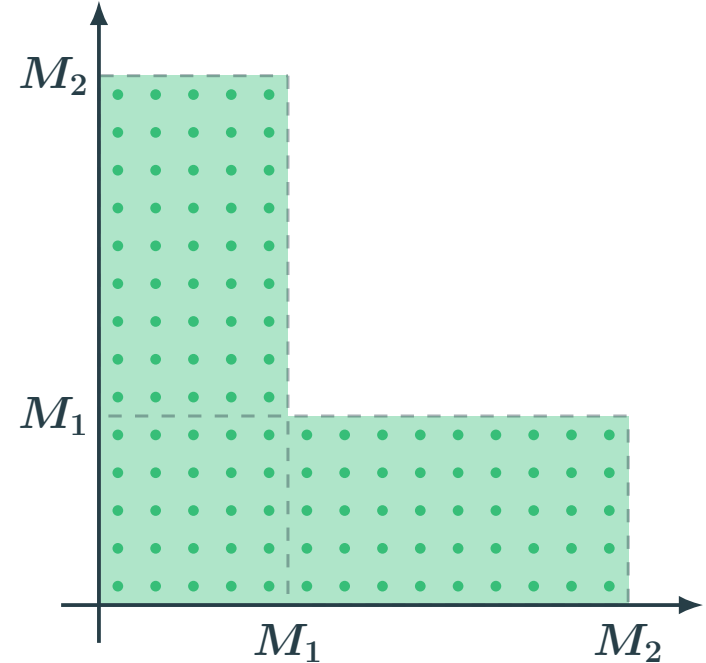
$$M_1 \ll M_2 \ll \dots \ll M_d.$$

Precisely,

$$M_1 = n \cdot n^{4^0}, M_2 = n \cdot n^{4^1}, \dots, M_d = n \cdot n^{4^{d-1}}.$$

How many thin configurations exist?

$$\begin{aligned} &\leq d! \cdot M_1 \cdot M_2 \cdot \dots \cdot M_d = d! \cdot (n \cdot n^{4^0}) \cdot (n \cdot n^{4^1}) \cdot \dots \cdot (n \cdot n^{4^{d-1}}). \\ &= d! \cdot n^d \cdot n^{\sum_{i=0}^{d-1} 4^i}. \end{aligned}$$



Bounding the Length of Coverability Runs

Consider the shortest coverability run $\vec{u} \xrightarrow{\pi} \vec{w}$, where $\vec{w} \geq \vec{v}$.

Split π at first “non-thin” configuration: $\vec{u} \xrightarrow{\rho} \vec{x} \xrightarrow{\tau} \vec{w}$.

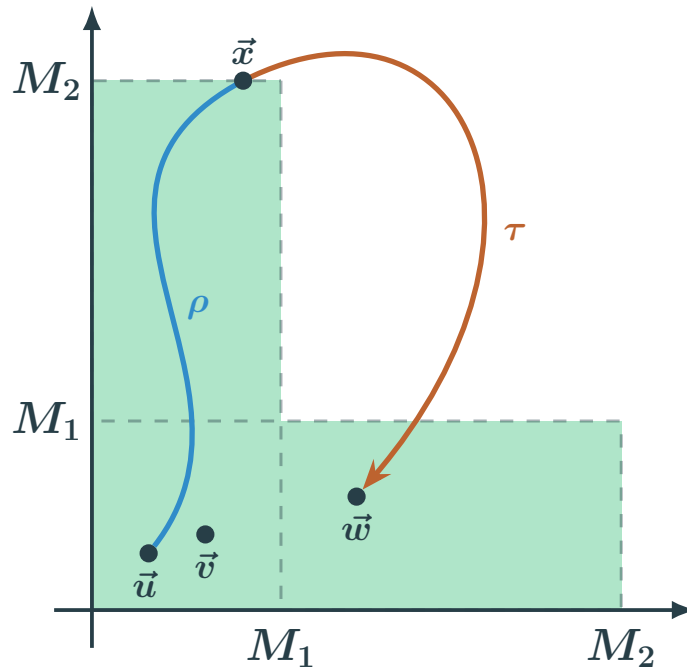
ρ is the *thin part* of the run, its length is bounded by the number of thin configurations.

Claim 1: $len(\rho) \leq d! \cdot n^d \cdot n^{\sum_{i=0}^{d-1} 4^i}$.

Proof idea: there cannot be any zero effect cycles in π .

τ is the *tail* of the run, at least one component had a large value at $\vec{x}$, so can then be ‘ignored’.

Claim 2: $len(\tau) \leq n^{4^{d-1}}$.



Using Rackoff's Inductive Technique

Claim 2: $len(\tau) \leq n^{4^{d-1}}$. (Proof by induction on d)

Sort the components $\vec{x}[1] \leq \vec{x}[2] \leq \dots \leq \vec{x}[d]$.

There exists $i \in \{1, \dots, d\}$ such that $M_i \leq \vec{x}[i]$.

Moreover, $M_i = n \cdot n^{4^{i-1}} \leq \vec{x}[i] \leq \dots \leq \vec{x}[d]$.

Example: $\vec{x}[1] < M_1$ but $\vec{x}[2] \geq M_2$.

Use induction, focussing just on the first $i - 1$ components.

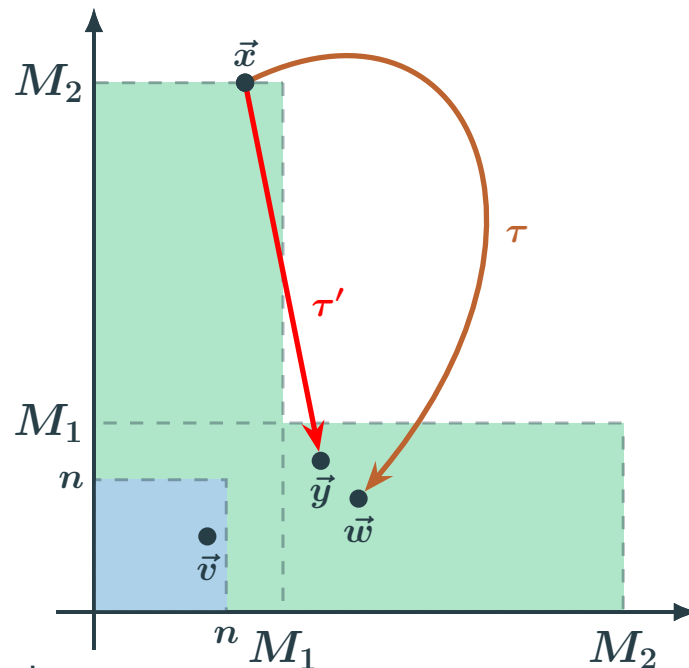
There is an alternative suffix τ' with $len(\tau') \leq n^{4^{i-1}}$ and

$$\begin{aligned} (x[1], \dots, x[i-1]) &\xrightarrow{\tau'} (\vec{y}[1], \dots, \vec{y}[i-1]) \\ &\geq (\vec{v}[1], \dots, \vec{v}[i-1]). \end{aligned}$$

We know that τ' has at least $-n \cdot (len(\tau') - 1)$ effect on each

of the remaining components. Fortunately, $(n \cdot n^{4^{i-1}}, \dots, n \cdot n^{4^{i-1}}) \leq (\vec{x}[i], \dots, \vec{x}[d])$.

So, $(\vec{x}[i], \dots, \vec{x}[d]) \xrightarrow{\tau'} (\vec{y}[i], \dots, \vec{y}[d]) \geq (n, \dots, n) \geq (\vec{v}[i], \dots, \vec{v}[d])$.



Proof of Main Theorem

Theorem: Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Proof: Let π be the shortest run witnessing coverability.

$$\begin{aligned} \text{len}(\pi) &= \text{len}(\rho) + \text{len}(\tau) \\ &\leq d! \cdot n^d \cdot n^{\sum_{i=0}^{d-1} 4^i} + n^{4^{d-1}} && \text{(By Claim 1 and Claim 2)} \\ &\leq 2 \cdot d! \cdot n^d \cdot n^{\sum_{i=0}^{d-1} 4^i} \\ &\leq n^{2^d} \cdot n^{\sum_{i=0}^{d-1} 4^i} && \text{(when } n \geq 2, \quad 2 \cdot d! \cdot n^d \leq n^{2^d} \text{)} \\ &\leq n^{4^d} && \text{(when } d \geq 1, \quad 2^d + \sum_{i=0}^{d-1} 4^i \leq 4^d \text{)} \\ &= n^{2^{2d}} = n^{2^{\mathcal{O}(d)}}. \end{aligned}$$

□

Algorithms for Coverability

Theorem: Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Corollary 1: Coverability in VASS can be decided in $2^{\mathcal{O}(d)} \cdot \log(n)$ space.

OPTIMAL!

Proof idea: Nondeterministically search through the configuration space, each configuration can be expressed with $2^{\mathcal{O}(d)} \cdot \log(n)$ bits.

Corollary 2: Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

CONDITIONALLY OPTIMAL!

Proof idea: Deterministically search through the configuration space.

Conditionally Optimal Time Bound

Corollary 2: Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

Theorem: Assuming the Exponential Time Hypothesis, coverability in VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Idea: Reduce detecting a 2^d -clique in a 2^d -partite n -vertex directed graph to coverability.

Conjecture (Exponential Time Hypothesis): 3-SAT with k -variables requires $2^{\Omega(k)}$ time.



Detecting whether there is a k -clique in a k -partite n -vertex graph requires $n^{\Omega(k)}$ time.

[Chen, Chor, Fellows, Huang, Juedes, Kanj, and Xia '05]

[Chen, Huang, Kanj, and Xia '06]

[Cygan, Fomin, Kowalik, Lokshtanov, Marx, Ma. Pilipczuk, and Mi. Pilipczuk '15]

Bounded Two-Counter Machines

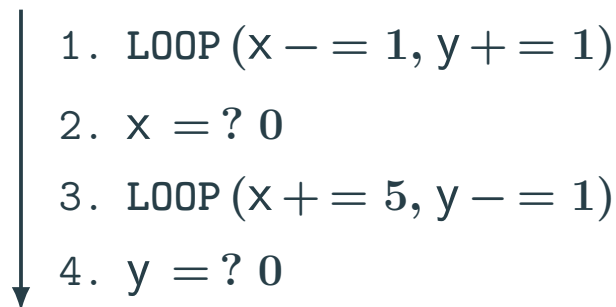
Idea: Reduce detecting a 2^d -clique in a 2^d -partite n -vertex directed graph to coverability.

First, reduce to coverability in a $n^{2^{\mathcal{O}(d)}}$ -bounded two-counter machine.

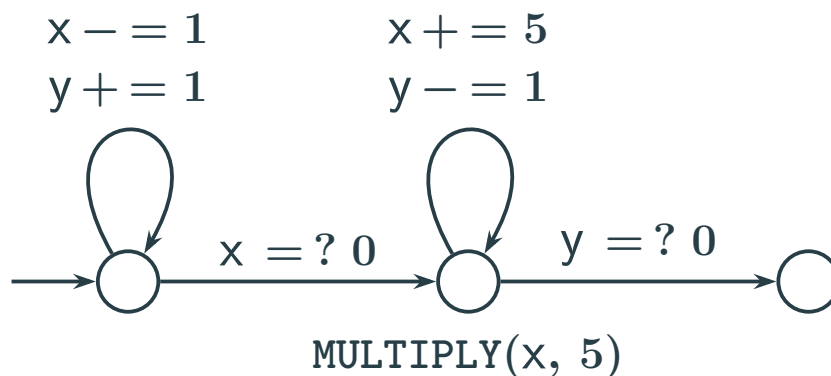
Then, simulate a $n^{2^{\mathcal{O}(d)}}$ -bounded two-counter machine using an $\mathcal{O}(n)$ -state $\mathcal{O}(d)$ -VASS.

An $n^{2^{\mathcal{O}(d)}}$ -**bounded two-counter machine** has two counters $x, y \in \{0, 1, \dots, n^{2^{\mathcal{O}(d)}}\}$ that can be added to ($x += 2$), subtracted from ($y -= 3$), and zero-tested ($x = ? 0$).

Pre: $x = x, y = 0$



Post: $x = x \cdot 5, y = 0$



Bounded Two-Counter Machines

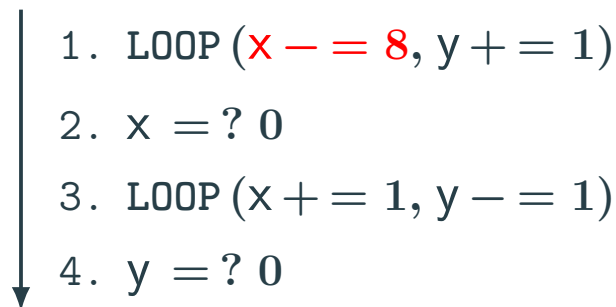
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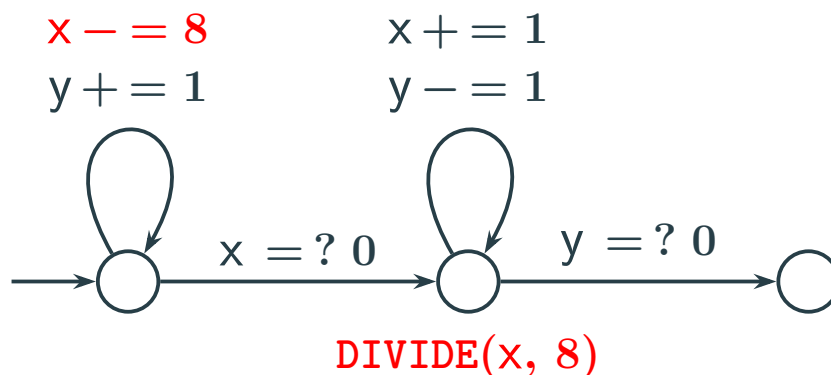
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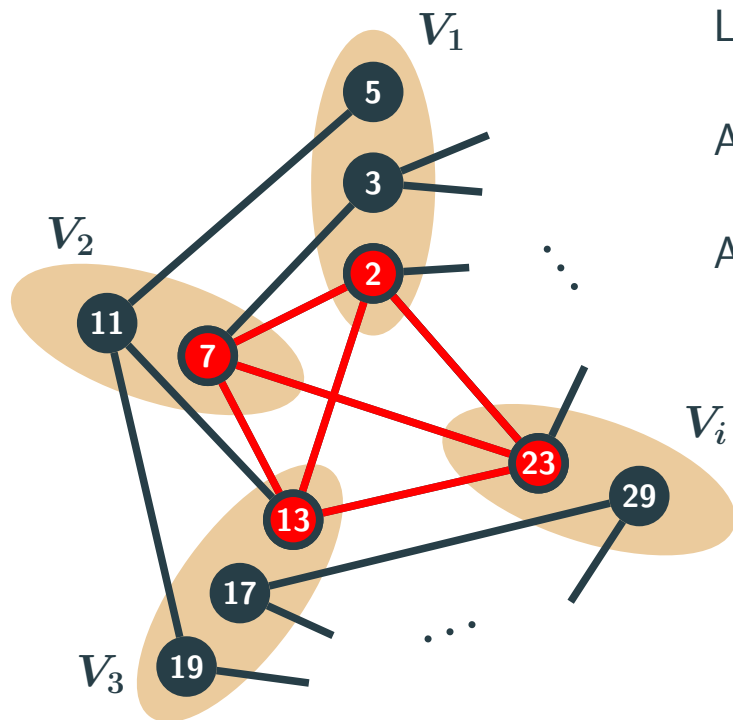
Pre: $x = x, y = 0$



Post: $x = x \div 8, y = 0$



Detecting Cliques using Divisibility Tests



Let $(V_1 \cup V_2 \cup \dots \cup V_k, E)$ be a k -partite n -vertex graph.

Associate the first n primes with the vertices.

A candidate k -clique is represented by a product of k primes.

Example: $c = 2 \cdot 7 \cdot 13 \cdot \dots \cdot 23$.

To check if v represents a clique, use divisibility tests to verify all nodes are adjacent.

Example: $(2 \cdot 7) | c?$ $(2 \cdot 13) | c?$ $(7 \cdot 13) | c?$...
 $(2 \cdot 23) | c?$ $(7 \cdot 23) | c?$ $(13 \cdot 23) | c?$

There exist $p_1 \in \text{Primes}(V_1)$, $\dots$, $p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) | p_1 \cdot \dots \cdot p_k \iff$ there exists a k -clique.

Bounded Two-Counter Machine Implementation

There ϵ
is an ϵ

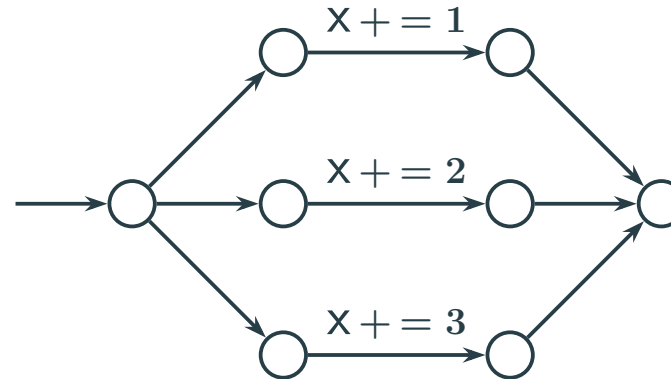
Guessing with Nondeterministic Branching

k , there
 k -clique.

Pre: $x = x$

- ↓
1. GUESS: $c \in \{1, 2, 3\}$
 2. $x += c$

Post: $x = x + 1$, or
 $x = x + 2$, or
 $x = x + 3$.



Bounded Two-Counter Machine Implementation

There exist $p_1 \in \text{Primes}(V_1), \dots, p_k \in \text{Primes}(V_k)$ such that for every pair $1 \leq i < j \leq k$, there is an edge $\{p, q\} \in (V_i \times V_j) \cap E$ such that $(p \cdot q) \mid p_1 \cdot \dots \cdot p_k \iff$ there exists a k -clique.

Part one: Guess a candidate clique.

Pre: $x = 1, y = 0$.

$\downarrow$
 1. GUESS: $p_1 \in \text{Primes}(V_1)$
 2. MULTIPLY(x, p_1)
 $\vdots$
 2k-1. GUESS: $p_k \in \text{Primes}(V_k)$
 2k. MULTIPLY(x, p_k)
 $\downarrow$
 Post: $x = p_1 \cdot \dots \cdot p_k, y = 0$.

This two-counter program terminates

$\iff$ there exists a k -clique.

Part two: Check the candidate is a clique.

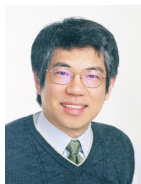
Pre: $x = p_1 \cdot \dots \cdot p_k, y = 0$.

$\downarrow$
 1. GUESS: $\{p_1, p_2\} \in (V_1 \times V_2) \cap E$
 2. DIVIDE($x, p_1 \cdot p_2$)
 3. MULTIPLY($x, p_1 \cdot p_2$)
 $\vdots$
 $<3k^2$. GUESS: $\{p_{k-1}, p_k\} \in (V_{k-1} \times V_k) \cap E$
 $<3k^2$. DIVIDE($x, p_{k-1} \cdot p_k$)
 $<3k^2$. MULTIPLY($x, p_{k-1} \cdot p_k$)
 $\downarrow$
 Post: $x = p_1 \cdot \dots \cdot p_k, y = 0$.

VASS can Simulate Bounded Two-Counter Machines

Counter bound of k -clique detecting two-counter machine: $\mathcal{O}(p_{\max}^k) \leq \mathcal{O}(n^k \log(n)^k) \leq \mathcal{O}(n^{2k})$.

Size of k -clique detecting two-counter machine: $\mathcal{O}(n^{11}) \leq \text{poly}(n)$.



Louis Rosier Hsu-Chun Yen

Lemma: In $\text{poly}(n)$ time, one can construct a $\mathcal{O}(\log(k))$ -VASS that can *simulate* an $\mathcal{O}(n^k)$ -bounded $\mathcal{O}(1)$ -counter machine of $\text{poly}(n)$ size.

[Rosier and Yen '85]

If we set $k = 2^d$, the $\text{poly}(n)$ -size two-counter machine for detecting 2^d -cliques is $\mathcal{O}(n^{2^d})$ -bounded.

$\implies$ In $\text{poly}(n)$ time, one can construct an $\mathcal{O}(d)$ -VASS for detecting 2^d -cliques.

Remark: Here, termination is coverability.

“Can I get to the end of the program with any (at least zero) value on each of the counters?”

Reducing to Coverability in VASS

Detecting 2^d -cliques in an n -vertex graph requires $n^{\Omega(2^d)}$ time under the Exponential Time Hypothesis.



Via divisibility tests of a product of primes encoding.

First, construct an instance of termination in a $\text{poly}(n)$ -size $\mathcal{O}(n^{2^d})$ -bounded two-counter machine.



Using Rosier and Yen's simulation lemma.

Then, in $\text{poly}(n)$ time, construct an instance of coverability in an $\mathcal{O}(d)$ -VASS.

Theorem: Assuming the Exponential Time Hypothesis, coverability in VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

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Corollary 2: Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$ time.

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Theorem: Assuming the Exponential Time Hypothesis, coverability in VASS requires $n^{2^{\Omega(d)}}$ time.

[Künnemann, Mazowiecki, Schütze, S-B, and Węgrzycki '23]

Thank You!

Presented by Henry Sinclair-Banks, University of Warwick, UK 

Verification Seminar in IRIF, Paris, France 

