Infinite Automata 2025/26

Lecture Notes 10

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Recall from Lecture 9, that we wish the prove the following theorem.

Theorem 9.3. Coverability in VASS is in EXPSPACE (regardless of whether the problem is encoded in unary or binary). Moreover, coverability in unary-encoded and binary-encoded d-VASS is in NL and PSPACE, respectively.

To prove Theorem 9.3, we will establish the existence "short" runs for coverability. Here "short" means that the runs are doubly exponential in length; i.e. if coverability holds, then there is a run of length at most $n^{d^{\mathcal{O}(d)}}$ (here n is the unary-encoded size of the instance and d is the dimension of the VASS). Assuming this result, it follows that there is a non-deterministic algorithm for coverability that uses $d^{\mathcal{O}(d)} \cdot \log(n)$ space. This places coverability in VASS in NEXPSPACE, and by Satich's theorem, NEXPSPACE = EXPSPACE. When the dimension d of the VASS is fixed, then $d^{\mathcal{O}(d)}$ is just some (potentially large) constant. This means that there is a non-deterministic $\mathcal{O}(\log(n))$-space algorithm for coverability in unary-encoded d-VASS. Accordingly, when the input is encoded in unary, the problem is in NL, and when the input is encoded in binary, the problem is in NPSPACE. Again, thanks to Savitch's theorem, NPSPACE = PSPACE. To establish Theorem 9.3, we will state and prove Lemma 10.1.

Given that a d-VASS can be simulated by a $(d+3)$-VAS (see Exercise 6.2), we shall work with VAS instead of VASS. We shall fix our attention on a d-VAS $V \subset \mathbb{Z}^d$, an initial configuration $\mathbf{s} \in \mathbb{N}^d$, and a target configuration $\mathbf{t} \in \mathbb{N}^d$. We use $\|\mathbf{v}\| = \max\{1, \|\mathbf{v}\|_\infty\}$ to denote a function that is not technically a norm (we want the size of a vector to always be at least 1). Let $n = \sum_{\mathbf{v} \in V} \|\mathbf{v}\| + \|\mathbf{t}\|$. Notice that the size of $\mathbf{s}$ does not contribute to the value of n.

Lemma 10.1. Let $L_d \coloneqq n^{(2d+2)^d}$. For any $\mathbf{s} \in \mathbb{N}^d$, if $\mathbf{s} \xrightarrow{*}_V \mathbf{t}'$ for some $\mathbf{t}' \geq \mathbf{t}$ then there exists a run π such that $\mathbf{s} \xrightarrow{\pi}_V \mathbf{t}''$ for some $\mathbf{t}'' \leq \mathbf{t}$ and $|\pi| \leq L_d$.

We will prove Lemma 10.1 by induction on the dimension d. The base case is $d = 0$. In a 0-dimensional VAS, the only possible configuration is the empty vector $\varepsilon \in \mathbb{N}^0$ and there is only one (trivial) run $\varepsilon \xrightarrow{*} \varepsilon$ which has length 1. This satisfies the conditions of the lemma as $L_0 = n^{1^0} \geq 1$.

For the inductive step, when $d \geq 1$, we shall assume that Lemma 10.1 holds for the dimension $d-1$. Let $\pi = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ be a run in V with minimal length such that $\mathbf{s} \xrightarrow{\pi} \mathbf{t}'$ for some $\mathbf{t}' \geq \mathbf{t}$. Note that $\mathbf{c}_0 = \mathbf{s}$, $\mathbf{c}_\ell = \mathbf{t}'$, and for every $j \in \{1, \ldots, \ell\}, \mathbf{c}_j - \mathbf{c}_{j-1} \in V$. Our goal is to prove that $|\pi| = \ell + 1 \leq L_d$.

Observe that all configurations $\mathbf{c}_i$ must be distinct from each other, else π could be trivially shortened by removing a zero-effect cycle.

Definition 10.2. *Small configurations.* In a d-VAS, we say that a configuration $\mathbf{c} \in \mathbb{N}^d$ is *small* if, for every $i \in \{1, \ldots, d\}$, $\mathbf{c}[i] < L_{d-1} \cdot n$.

Let $m \in \{0, \ldots, \ell\}$ be the first index such that $\mathbf{c}_m$ is not small, otherwise let $m = \ell + 1$. We shall decompose π about the m-th configuration $\pi = \pi_{\text{small}} \cdot \pi_{\text{tail}}$, where $\pi_{\text{small}} = (\mathbf{c}_0, \ldots, \mathbf{c}_{m-1})$ and $\pi_{\text{tail}} = (\mathbf{c}_m, \ldots, \mathbf{c}_\ell)$. Note that it is possible for π_{small} or π_{tail} to be an empty run. We will now analyse the lengths of π_{small} and π_{tail}.

Claim 10.3. $|\pi_{\text{small}}| \leq L_{d-1}^d \cdot n^d$.

Proof. We know that $\pi_{\text{small}} = (\mathbf{c}_0, \ldots, \mathbf{c}_{m-1})$ consists of distinct configurations. By definition of m, we know that $\mathbf{c}_0, \ldots, \mathbf{c}_{m-1}$ are all small configurations. Thus the length of π_{small} is bounded above by the total number of small configurations and there are $(L_{d-1} \cdot n)^d$ many small configurations. $\square$

Claim 10.4. $|\pi_{\text{tail}}| \leq L_{d-1}$.

Proof. Here we will use the inductive assumption. Let $i \in \{1, \ldots, d\}$ be an index such that $\mathbf{c}_m[i] \geq L_{d-1} \cdot n$ (such an index exists because $\mathbf{c}_m$ is not small). We shall consider a covering run in the $(d-1)$-VAS V_{d-1} that is defined by projecting away the i-th coordinate from every transition and the initial and target vectors. Precisely, let

$$\begin{aligned}\mathbf{s}_{d-1} &\coloneqq (\mathbf{c}_m[1], \ldots \mathbf{c}_m[i-1], \mathbf{c}_m[i+1], \ldots, \mathbf{c}_m[d]);\\ \mathbf{t}_{d-1} &\coloneqq (\mathbf{t}[1], \ldots \mathbf{t}[i-1], \mathbf{t}[i+1], \ldots, \mathbf{t}[d]); \text{ and}\\ V_{d-1} &\coloneqq \{(\mathbf{v}[1], \ldots, \mathbf{v}[i-1], \mathbf{v}[i+1], \ldots, \mathbf{v}[d]) : \mathbf{v} \in V\}.\end{aligned}$$

Since $\mathbf{c}_m \xrightarrow{\pi_{\text{tail}}} \mathbf{t}'$ and $\mathbf{t}' \geq \mathbf{t}$, we know that in V_{d-1}, there is an equivalent run (that takes the same transitions as π_{tail} with the i-th coordinate projected away) from $\mathbf{s}_{d-1}$ to a configuration $\mathbf{t}'_{d-1}$ where $\mathbf{t}'_{d-1} = (\mathbf{t}'[1], \ldots, \mathbf{t}'[i-1], \mathbf{t}'[i+1], \ldots, \mathbf{t}[d]) \geq \mathbf{t}_{d-1}$. By the inductive assumption, we know that there exists a run π_{d-1}: $\mathbf{s}_{d-1} \xrightarrow{\pi_{d-1}}_{V_{d-1}} \mathbf{t}''_{d-1}$ where $\mathbf{t}''_{d-1} \geq \mathbf{t}_{d-1}$ and $|\pi_{d-1}| \leq L_{d-1}$.

Now, we shall lift this run π_{d-1} back to V and argue that it is a covering run from $\mathbf{c}_m$ to a configuration that is at least $\mathbf{t}$. On coordinates $j \in \{1, \ldots, i-1, i+1, \ldots, d\}$ the run π_{d-1} acts identically and reaches $\mathbf{t}''_{d-1}[j] \geq \mathbf{t}_{d-1}[j] = \mathbf{t}_{d-1}[j]$. Thus, we only need to consider the i-th coordinate. Recall that $\mathbf{c}_m[i] \geq L_{d-1} \cdot n$. In the worst case, the run π_{d-1} projected to V has the most negative possible effect on the i-th coordinate; we know that, for every $\mathbf{v} \in V$, $\mathbf{v}[i] \geq -n$. Thus, starting from $\mathbf{c}_m[i]$, we know that the effect of π_{d-1} on the i-th coordinate is at least $-n \cdot (L_{d-1} - 1)$. Note that if π_{d-1} has length L_{d-1}, then the run takes at most $L_{d-1} - 1$ many transitions. At the end of the run, we know that the i-th counter will have value at least

$$\mathbf{c}_m[i] - n \cdot L_{d-1} + n \geq L_{d-1} \cdot n - n \cdot L_{d-1} + n = n \geq \mathbf{t}[i].$$

This means that the lifted run, in V, starting from $\mathbf{c}_m$ leads to a configuration $\mathbf{t}''$ for some $\mathbf{t}'' \geq \mathbf{t}$ and has length at most L_{d-1}.

Lastly, we conclude the proof of this claim by recalling that the original run $\mathbf{s} \xrightarrow{\pi} \mathbf{t}'$ was selected such that π had minimal length. This means that the length of the suffix π_{tail} is not greater than the length of the lifted run π_{d-1}, hence $|\pi_{\text{tail}}| \leq |\pi_{d-1}| \leq L_{d-1}$. ☐

To conclude the proof of Lemma 10.1, we shall combine Claim 10.3 and Claim 10.4.

$$\begin{aligned}|\pi| = |\pi_{\text{small}}| + |\pi_{\text{tail}}| \leq L_{d-1}^d \cdot n^d + L_{d-1}^d &= n^{(2d)^{d-1}} \cdot n^d + n^{(2d)^{d-1}}\\ &\leq 2n^{(2d)^{d-1}+d}\\ &\leq n^{(2d)^{d-1}+d+1}\\ &\leq n^{(2d)^{d-1}+2d}\\ &\leq n^{(2d)^d}\\ &< n^{(2d+2)^d} = L_d.\end{aligned}$$

We now have a very strong understanding of the complexity of coverability in VASS. In the following table, the red text refers to the lower bound and the blue text refers to the upper bound. We have not proved that coverability in VASS is EXPSPACE-hard when the dimension is not fixed.

Dimension	**Unary encoding**	**Binary encoding**
1	NL-complete Theorem 1.8, Theorem 9.3	NL-hard and in P Theorem 1.8, Exercise 9.5
≥ 2	NL-complete Theorem 1.8, Theorem 9.3	PSPACE-complete Theorem 1.8, Theorem 9.3
not fixed	EXPSPACE-complete Lipton (1976), Theorem 9.3	